R E P O R T   R E S U M E S

ED 011 081   24

TEACHING OF ADVANCED MATHEMATICAL CONCEPTS TO CULTURALLY DISADVANTAGED ELEMENTARY SCHOOL CHILDREN.

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REPORT NUMBER CRP-S-640-65   PUB DATE   66
REPORT NUMBER BR-5-8396
EDRS PRICE MF-$0.18 HC-$3.56   89P.

DESCRIPTORS- *DISCOVERY LEARNING, *CULTURALLY DISADVANTAGED, *MATHEMATICS INSTRUCTION, *ELEMENTARY EDUCATION, *LEARNING MOTIVATION, MATHEMATICAL CONCEPTS, ADVANCED PROGRAMS, BERKELEY

THE SUCCESS OF DISCOVERY MATHEMATICS TEACHING IN THE ELEMENTARY SCHOOL WAS TESTED OVER A 1-YEAR PERIOD. THE PROJECT WAS INTENDED TO SEE IF A TRAINED MATHEMATICIAN WORKING AT AN ELEMENTARY SCHOOL WITH DISADVANTAGED CHILDREN COULD (1) MOTIVATE THE CHILDREN TO BE INTERESTED IN SCHOOL WORK BY INTERESTING THEM IN MATHEMATICS AND (2) COMMUNICATE WITH THEM BY TEACHING THEM TO DO MATHEMATICS AT A LEVEL FAR ABOVE GRADE LEVEL, EVEN TO HIGH SCHOOL LEVEL. ACHIEVEMENT OF THE FIRST OBJECTIVE, WHILE NOT SCIENTIFICALLY PROVED, WAS STRONGLY SUPPORTED (HYPOTHESIS) BY THE HIGH LEVEL OF EXCITEMENT WHICH EXISTED IN THE FOUR CLASSES STUDIED. IT WAS CONCLUDED THAT ADEQUATE MEASUREMENT OF AN INCREASE OF INTEREST CAUSED BY THE MATHEMATICS PROGRAM WOULD PROBABLY REQUIRE STUDY OF A CLASS FOR SOME TIME BEFORE PRESENTING IT WITH A DISCOVERY TEACHER. THE CONTROLS NECESSARY TO CONCLUSIVELY PROVE THE FIRST HYPOTHESIS COULD NOT BE ESTABLISHED. ACHIEVEMENT OF THE SECOND OBJECTIVE WAS FULLY BORNE OUT BY THE DATA. EACH CLASS MADE SIGNIFICANT PROGRESS WITH MANY MATHEMATICAL NOTIONS WHICH UNDERLIE ADVANCED MATHEMATICS. (TC)

5-8396

Teaching of Advanced Mathematical Concepts to Culturally Disadvantaged Elementary School Children

Cooperative Research Project No. S-640-65

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1966

The Research reported herein was supported by the Cooperative Research Program of the Office of Education, U. S. Department of Health, Education and Welfare.

ED 011081

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1. Research Problem

The current social upheaval in the United States has made educators increasingly aware that of the major problems facing them today, none is more important than that of bringing the culturally disadvantaged child into the mainstream of American education. The gap between the main culture in the United States today and minority subcultures poses two serious problems: (1) the problem of motivation and (2) the problem of communication.

Motivating disadvantaged children in the classroom is hampered by the fact that these children often have strong negative feelings about themselves. They are unlikely to be interested in schoolwork as long as they are unsuccessful, and success is rare for those who already have begun to feel that they cannot succeed. A past approach has been for the schools to stress subjects such as art, music and sports on the grounds that disadvantaged children can be successful in these areas. However, society places more and more value on excellence in strictly academic subjects. Accordingly, stressing non-academic subjects is a blatant sign of disrespect for disadvantaged young people --- such stress is bound to reinforce their negative feelings about themselves.

Mathematics has always been one of the most prestigious disciplines in our society. Second graders are able to relate algebra to the exploits of the astronauts. The rigor and depth of mathematics demands respect for the subject even from those who do not care for it. Thus, to offer mathematics (not arithmetic)[x] to young children

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x It is important to realize that throughout this report mathematics is being distinguished from the traditional study of techniques in arithmetic. Mathematics might well be called the study of how to find out which sentences about mathematical structures are true. The heart

is to pay them a compliment and show respect for them. This sign of respect can go a long way toward erasing the negative self-image which cripples motivation.

Clearly, children must like what they study, if they are going to be motivated to continue. Mathematicians assert almost unanimously that mathematics if fun. In fact, the general public has traditionally found mathematics to be a source of recreation. Those who claim to dislike it are usually people who have had bad experiences with it in school. Children still in elementary school, especially in the primary grades, have not yet been scarred in this way. Thus, it seems reasonable that a mathematician can impart his own interest and enthusiasm for mathematics to children. Moreover, he can make it fun for them by spontaneously choosing from a broad range of subjects and by turning problem-solving into puzzle-solving and play. Because of the depth and wide scope of mathematics, considerable training in the subject would be required to teach mathematics in this manner. Such training could not be expected of an elementary school teacher.

The second problem facing disadvantaged school children, is the problem of communication. In no area is the culturally disadvantaged child more educationally deprived than in his verbal abilities. It is disheartening to observe a child slip further and further below grade level in reading as he progresses through school and to realize that he is slowly, but very surely, losing his chance for an education. Such a child is often way behind many of his middle-class counterparts upon entering kindergarten. This situation calls for drastic measures to remedy his reading problems before they become too advanced. Otherwise, his future

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of mathematics is the development of new notions and reasoning about them, so as to accomplish the task. This report is concerned with the feasibility of introducing the science of mathematics to disadvantaged children.

learning will be impossibly difficult and burdened with a growing sense of failure and hopelessness.

A startling and novel approach to the problem has been developed by William Johntz, a high school teacher in Berkeley, California, who has been teaching mathematics to elementary school children since 1963. The core of his idea is that the disadvantaged child is not behind his more privileged middle-class counterpart in mathematical experience when both begin school. Both have had only the most primitive mathematical experience. Neither can be said to lag significantly behind the other, since both are just starting. In other words, the disadvantaged kindergarten or first-grade child is not disadvantaged in the area of mathematical knowledge. The possibility of communicating mathematically with him could be exploited when he is in kindergarten and carried through until he finishes high school.

Furthermore, in the years which follow, an understanding of mathematical concepts is not developed outside school, in the home or otherwise, in very many children of any socio-economic class. Most mathematical learning during the school years actually occurs in school. Thus, culturally disadvantaged children will not be at a disadvantage in mathematics, if, by special effort on the part of the schools, an appropriate mathematics program could be worked out covering all the school years.

Special effort could even lead to the otherwise culturally disadvantaged child becoming mathematically advantaged. The old myth that disadvantaged children cannot really learn an academic, abstract discipline would be dramatically shattered.

If mathematical communication is thus not excluded, the manner of establishing this communication is still a great problem. We have noted that an elementary school teacher cannot be expected to present real mathematics to children and that a specialist is required. The question

arises whether a mathematically trained person who is not a regular elementary school teacher can accomplish the task, whether he is suited to present his ideas to children and whether there are techniques for him to use which will be successful with disadvantaged children.

An answer to this question is provided by Johntz. He has adapted the "discovery" technique of teaching mathematics,[x] as developed by Robert B. Davis of the Madison Project in his book Discovery in Mathematics, to his own elementary school teaching in Berkeley (this adaptation involves significant alterations). Moreover, he suggests that this technique is well suited to mathematicians and that the active and continuous oral communication between the teacher and the class entailed by it is an ideal way of minimizing the verbal handicaps of disadvantaged children. These children, who do not read well and who have difficulty listening to direct explanations, will participate in a dialogue, and when they have understood a question, they will become excited and highly motivated to find the answer. Thus communication begins via mathematics.

These rather general speculations on the problems of motivation and communication and the possible role of mathematics in their solution lead to the formulation of a research problem.

(I) Can a trained mathematician who is not an elementary school teacher teach mathematics to culturally disadvantaged elementary school children in such a way as to motivate them, both in mathematics and in other areas, and also to communicate with them, so that they in fact learn mathematics? What will be the results if a mathematician spends one classroom period (separate from the usual arithmetic period) each day of the week for one

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x It should be noted that henceforth in this report, we have something quite definite in mind when we refer to "the discovery technique of teaching mathematics."

school year teaching mathematics by the discovery technique with the regular teacher in the room to maintain control? What are the effects of varying certain aspects of this situation? Which mathematical topics are the most successful and how can they be best presented?

The present report and the work it describes are a contribution to the study of this problem. In the following pages specific hypotheses will be stated. The teaching which was carried out to test them will be described, and the extent to which they are borne out will be discussed.

A precise solution to problem (I) could be obtained if the following problem were solved. This problem is, moreover, more amenable to a precise solution.

(II) What will be the result of providing the culturally disadvantaged child with extra mathematical instruction (as envisaged in (I)) from grades K to 12? Will the child become motivated to succeed in school, will he attain a high level of competence in mathematics, and will he qualify for college?

In my opinion (II) is a more significant question than (I). Whereas one successful year in school sometimes can cause a child to develop a pattern of improvement, the causes of failure in school of culturally disadvantaged children are so deeply rooted in the environment which they enter at birth and in which they live during their entire schooling, that we might as well postulate that a successful compensatory education program must encompass all schooling from grades K - 12. Thus, a positive answer to problem (I) urgently calls for a large scale attack on problem (II). A longitudinal study of this problem extending over a period of several years must be undertaken now.

2. Related Research

Closely related to the present research is the work of what might unofficially be termed " the Johntz project." Since 1963 Johntz has been teaching modern algebra and coordinate geometry in predominantly Negro Berkeley elementary schools.[x] Since Fall 1965, eight mathematicians, mostly graduate students at the University of California in Berkeley, have been teaching mathematics to classes in these schools in association with Johntz. The number of these teachers will be higher during the school year 1966-1967. Moreover, several of the original eight teachers are starting similar projects in Oregon, Alabama, and North Carolina. The Johntz project is planning to expand from Berkeley to neighboring communities. A start has already been made in Richmond, just north of Berkeley.

All of the teachers use the discovery technique. The classes have been similar in composition to Johntz' original classes. A variety of mathematical topics have been presented, most of which are considered too advanced for elementary school children. As one of the eight mathematicians who first began to work with Johntz in Berkeley, I taught under essentially the same circumstances; those circumstances will be described later in detail.

Since the expanding Johntz project now proposes to offer special mathematics teaching from grades K - 9 to selected classes which will be held together over the years in several schools in the Berkeley area, an opportunity already exists to solve problem (II). A long-range study of these classes is called for.

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x In Berkeley, those children who are culturally disadvantaged (according to the definition to be given) come mostly from the Negro community in West Berkeley and attend one of four elementary schools.

3. Hypotheses

Before actually stating hypotheses, it is necessary to describe a teaching situation. The hypotheses will refer to this situation.

Teaching situation:

A mathematician at least at the bachelor's degree level spends at least four days a week during an entire school year in an elementary school classroom. He spends 35 - 50 minutes with the class each day. This period is separate from the class's usual arithmetic period. The pupils are mostly culturally disadvantaged, although of above average ability relative to their school. The regular teacher is always present, primarily to assume most of the burden of maintaining control. The mathematician teaches various non-standard and sometimes advanced mathematical topics by the discovery technique.

Two definitions need to be made for clarification. The term "culturally disadvantaged child" generally refers to a child who, as a consequence of being in a particular culture, suffers educational deficiencies, such as inadequate verbal experience, no commitment to classroom learning (the child's peer group usually reinforces an anti-intellectual attitude), a negative self-image, especially where intellectual abilities are concerned, and emotional disturbances resulting from an unhappy home life. It is not difficult to find children who suffer from some or all of these deficiencies. However, I do not feel at ease with a definition of the term "culturally disadvantaged child" which involves mention of such deficiences, if for no other reason than that I am not a sociologist or psychologist trained to speak of them.

Accordingly, in this report I shall have in mind the child who attends school or lives in an area of his community where schoolwork and achievement are significantly below grade level relative to the community as a whole.

The discovery technique to which I refer has as its central feature question asking. In a typical teaching period the mathematician begins with a series of easy questions directed to the entire class. Initially the questions may not even be closely related. They can be simple questions serving as a review of areas where the concepts already have been learned. He calls on pupils in various parts of the room and makes certain that the less able students answer an easy question. Sometimes gradually, and sometimes suddenly, the questions focus on a major problem, and the students become extremely anxious to know the answer. Although some information and explanation is offered by the teacher to define the mathematical situation and clarify the problems, he never reveals any of the answers to his questions. He affirms correct responses, but usually responds to incorrect ones by asking other questions which illustrate that another answer is needed. The students come closer to solving the main problem by themselves supplying partial solutions, solutions in particular cases, or solutions to related simpler problems. Suddenly a student "discovers" the mathematical relationship underlying the entire line of inquiry. Ideally this discovery is like a revelation. Frequently it is simply a feeling of comprehending. In any event, the sense of understanding is the student's reward for having participated. Teacher praise is not comparable to this reward and is usually handed out in a casual way. In fact, children seem to resent any praise which appears designed to lure them into accepting the teacher's standards in place of their own. They do not resent the impartial standards of mathematics.

After a few students make the discovery, the teacher tries to get them to convey it to the rest of the class, so that as many students as possible will be swept up in

the initial discovery of the solution to the problem. Once the problem is set aside, the teacher should return to it again after a few days (the " spiral" technique). When he does, more students should be led to make the discovery for themselves, until most of the class understands the original problem.

The above description outlines the discovery technique. Various special techniques may be used within this framework. For example, in order to change the pace and also to spot check the students understanding, it is often wise to ask the class to write down the answer to a question and then go quickly up and down the aisles checking answers. Brief individual help is then possible. Another special technique is that of developing ideas contained in wrong answers. Frequently the ideas are very creative, and the mathematical systems which grow out of "wrong answers" may be quite interesting, although they are at variance with the usual system. As another technique, the assigning of one or more "homework" problems is a good practice.[x]

We now set forth the hypotheses. The first two are of a general nature and are the main hypotheses which the research was designed to test. The further ones are more specific and are conjectures about what happens if certain aspects of the teaching are varied within the framework of the basic teaching situation described above. Because they were formulated during the research and not before, they are in a sense also conclusions.

As a consequence of the mathematics instruction:

1. Motivation of the children shows an increase. Interest and excitement are at a high level.
2. The children learn mathematics. They learn to deal with sophisticated concepts.

x It is commonly assumed that disadvantaged children will do no homework. However, the interest in mathematics classes can become so high that the homework record is fairly good.

2. (continued)

They understand and can do mathematics at an advanced level involving subject matter normally presented several grades later, and even subject matter related to mathematics not normally taught in the schools.

It is further conjectured that:

3. For effective teaching with the discovery technique it is essential that the class be homogeneously grouped by ability.
4. The better the regular classroom teacher is in maintaining control, the more attentive and responsive the class is to the mathematics teacher. In fact, if the regular teacher has poor control, effective mathematics teaching is not possible.
5. Firm measures to establish control should be used immediately, if a control problem starts to develop. The mathematics teacher must be prepared to participate. Rather than develop resentment of discipline which blocks learning, the children want firmness, and after being disciplined they quickly snap into a frame of mind appropriate for learning. On the other hand, resentment can develop, if a control problem is allowed to linger on.
6. The optimum class size for discovery teaching of mathematics is about 20. Class size should not lie outside the range 15 - 25.
7. If control is excellent, attention is good and interest is high, the length of the mathematics period can be as much as 50 minutes. The period should not be shorter than 35 minutes. The proper development of ideas cannot occur in less time.

8. If the amount of structuring by the teacher of the mathematical ideas and discussion is held to a necessary minimum, success in motivating the children, in increasing their self-confidence and self-reliance, and in making mathematics enjoyable for them is maximized.

4. Procedure, Analysis of the Data and Findings

The experimental procedure consisted in creating the teaching situation to which the foregoing hypotheses apply and observing what took place. My own mathematical background is a part of the situation, so I will describe the relevant features of it. I received a B. S. in Mathematics in 1959. From 1960 to 1965 I was a graduate student in mathematics at the University of California in Berkeley, specializing in Logic and the Foundations of Mathematics. In 1963 I received an M. A. in Mathematics. I had some experience as a teaching assistant in calculus courses and in a summer mathematics program for high school students. As preparation for the teaching of this project, in May 1965 I observed a fourth grade mathematics class taught by Johntz and actually taught this class for three weeks.

The mathematical background just described is far more extensive than necessary for discovery teaching in elementary schools. Most anyone with a bachelor's degree in mathematics, a necessary minimum of mathematical self-confidence, and a desire to teach children, ought to be qualified. However, my own more advanced knowledge of the foundations of mathematics was very useful to me. It aided me in judging the relevance of certain topics to the mathematical development of the children.

The three week period of "practice teaching" was also more than needed. Some of the mathematicians who joined the Johntz group merely observed Johntz at work a few times and then started their own classes. This lack of need for "training" may seem startling to some. However, the mathematics teachers are highly trained --- they are mathematicians! The content of the mathematics dictates the content of the teaching. Given that a person

desires to teach mathematics to children, the question whether he has had teacher training or studied education is utterly insignificant in comparison with the question whether he understands the mathematics he proposes to teach.

From October 1, 1965, to May 15, 1966, I taught mathematics to four classes at Lincoln Elementary School in Berkeley, California. The grade levels were two, three, four and six. Lincoln, one of the four West Berkeley elementary schools, has nearly 100% Negro enrollment. It lies in the most economically depressed area of Berkeley, and for many years its pupils scored (on the average) the lowest in Berkeley on standardized aptitude and achievement tests.[x] Schoolwork and achievement at all four of the West Berkeley elementary schools are significantly below grade level relative to Berkeley norms, when measured by written, objective tests. Accordingly, the children in my classes were culturally disadvantaged in terms of the operational definition given earlier. Interestingly enough, Berkeley elementary school pupils score way above national norms on the Stanford Achievement Test, and Lincoln pupils score just a shade below the national norms.

Particular features of the four classes I taught at Lincoln are relevant to the teaching situation. Each class was chosen for the experiment because it was an above average class at its grade level. An effort was made to have the classes homogeneous in ability, and this attempt was mostly successful. Inevitably, each class contained a few children well below the level of their class as a whole.

The primary grade classes (2-3) were wholly self-contained. They met the entire day in one room with one teacher (or with specialists, including myself, who taught them for certain periods with the regular teacher observing).

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x Lincoln has begun to climb in the last two years because of vigorous and creative leadership by its principal.

The upper grade classes (4-6) were partially self-contained. Each had a home room with a regular teacher. They spent about 3/5 of the school day with the regular teacher (or with specialists, including myself). The reading and arithmetic periods for the upper grades at Lincoln were the first two morning periods. Each pupil met with a group working at exactly his level (at the beginning of the school year the grouping was done on the basis of objective tests in reading and arithmetic), and this group usually included several pupils from other home rooms.

My second grade class was a superior one. The year before I taught it, this class participated in a special reading program. The children had a sense of being special before I came to them. They reflected this attitude in their response to the mathematics instruction, for they showed more self-confidence and independence in thinking than any other group I encountered at Lincoln. The second grade teacher exercised very strong control. The children liked her, but did not dare misbehave. I never once had to participate in maintaining control. I met with this class 35 - 40 minutes each school day. Class size was about 25. There were two distinct ability levels represented about equally in this class, but the slower half were sufficiently eager to still try to answer difficult questions.

The third grade class was very large the first semester. It had 33 pupils. About 10 went to another class second semester, leaving a class of 23 pupils. Control was very good both semesters, but communication was hampered by the class size first semester. Whereas observers felt the degree of attentiveness was quite high, I feel it did not always measure up to the level which one can expect in discovery teaching of mathematics.[x]

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x "Intense concentration" is the appropriate phrase to describe the degree of attentiveness one normally achieves.

Reduction of class size represented an improvement. The mathematics instruction lasted 35 minutes with the third grade.

The fourth grade class had from 25 - 29 pupils during the year. We met about 45 minutes each afternoon. The teacher was a man, the only man among the four teachers with whom I worked, and he could keep his class under as strict control as he chose. As a third grade class, this group had had Johntz as a discovery mathematics teacher. Accordingly, they had some mathematical sophistication when they first came to me. At times the prior experience of the class with Johntz created the problem of co-ordinating my approach with his in order to take advantage of this experience. One discovery teacher who succeeds another should certainly consult with or get an outline of material covered from the former. A few children in this class were enough below the ability level of the others that they did not understand the mathematics.[x] Such children must be recognized at the beginning of the year and must be given some individual attention. If such attention is provided at the beginning of the year, many will blossom as time passes and do quite well (and become very enthusiastic). If it is not given, these same children will "drop out" mentally and participate in body but not spirit for the remainder of the year.

During the last month or so of teaching, a book was used, and I worked individually with the children. Otherwise, all teaching was by the discovery technique.

The sixth grade class was another outstanding one. They were well above the level of Lincoln's other sixth grades. They had discovery teaching of mathematics from Johntz for their last month in the fifth grade. They were

x Furthermore, some of these were not in the class as third graders. They probably should have gone to a different class.

enthusiastic about mathematics as a result. Some of the boys had even come to take the subject seriously as an important part of their future education. Although many of the girls were equally bright, they did not feel so strongly that mathematics was their subject. During the entire year a control problem existed with the sixth grade. The first two weeks of the year their teacher was a man. He was transferred to an administrative position and replaced by a woman who had to struggle to attain a minimum degree of control. I began my instruction after she had started. It developed moreover that the mathematics instruction was replacing a full period of physical education which had been scheduled for the same time of day, and that the children were aware of this fact. Initially I was severely hampered because many of the children (with the exception of the brightest boys) openly indicated that they did not want the "algebra!" [x] I had to cope with the resulting hostility when it was in a sense forced upon them. Moreover, I had to involve myself in the control problem, since the regular teacher was having difficulty. After some time a mutual adjustment was made. Most students participated in the mathematics. However, control was always a bit shaky. It was almost impossible to do any individual work with a pupil, for the others immediately got out of hand. As a result this class had two or three mental "dropouts" who followed very little the entire year. On the other hand, the class was bright enough and sufficiently interested (after the first month) to learn some rather advanced mathematics. Good control of a discovery class does not require perfect order. Sometimes a good class will get very boisterous in its mathematical enthusiasm. Good control requires the ability to call for perfect order at any time and get it. It will be seen that there

x My classes were referred to as "algebra" although other mathematics was involved.

were times when perfect order and attentiveness were needed, in order for the class to understand certain ideas, and that only a few students did understand them, because this order could not be gained. I believe that all of the difficulties could have been avoided, if certain measures had been taken at the beginning of the year. The conflict with physical education never should have arisen. The children should have known from the first day of school that "algebra" would be every day at a certain period. Also a systematic plan to maintain control should have been developed with the regular teacher at the outset.

I believe in general that firm discipline, applied with absolutely no tolerance of minor infractions, straightens out the children, providing it is done before the control problem develops to any great extent. Furthermore, resentment of discipline is transitory, for in many ways the children want it and expect it. Perhaps in middle-class schools the children's upbringing might be such that firm discipline would be resented. Robert B. Davis of the Madison Project feels that it interferes with discovery teaching (see Discovery in Mathematics). However, the pupils in the Madison Project are middle-class. Of course, techniques of "discussing" rules and discipline with the children are needed too. Whereas this approach to the control problem was initially distasteful to me, I now accept it as being necessary to obtain the respect of the children and have a good teaching relationship with them. Once control is established, it is possible to allow quite a bit of freedom in behavior.

Class size was about 29 first semester and 25 second semester (several children transferred to another school and a couple of new ones entered the class). The class lasted about 45 minutes. The ability level was fairly uniform. Two or three were significantly below the group and about three were significantly above.

For about two weeks second semester a book was used, and I worked individually with many students. Otherwise, all teaching was by the discovery method.

After having this introduction to the classes, we turn to the main part of the report: a description of the mathematics taught and the response of the students. Hopefully, others who wish to try teaching mathematics by the discovery approach will find this material useful. As discovery mathematics programs start up in more and more schools across the nation, the need will grow for descriptions of mathematical topics (and the manner of presentation) which have been successful. Whereas the goal of this report is to evaluate my teaching last year, the report may also serve as a source of mathematical ideas for use in the schools.

Since most topics which I presented were dealt with at different levels in each of my classes, whenever a given topic is discussed in the sequel, the manner of presentation to each of the four classes and the responses of the classes will be handled all at once. Accordingly, it will be easy to compare the levels of understanding of the topics achieved by the classes. The last few topics are more advanced than the others and were not covered in the primary grade classes.

There is a standard introductory sequence which Johntz has developed. Every discovery class I know of has started this way for the simple reason that this sequence has been highly successful each time tried and that the mathematics involved forms an excellent foundation for much more mathematics. The topic is the addition of integers (the set of integers contains 0 and all positive or negative whole numbers), sometimes referred to as "addition of signed numbers." Perhaps some day addition of signed numbers will be treated in the schools as more basic than subtraction.

In any event, I believe that the algebra[x1] of signed numbers should be covered in the primary grades.

The opening lessons will now be described.[x2] A bead frame with red and yellow mushroom shaped beads which fit together in stacks on the frame served as a teaching aid. At the beginning of the first day the class was extremely curious as to what "algebra" was going to be. One should capitalize on the drama of the moment. Thus, after putting my name on the blackboard and pronouncing it, I held up a red bead and said: "Here is one red bead." Then I held up two red beads and asked the class what I was holding (2 red beads). I put "R_1" on the blackboard and said that it stood for one red bead and should be read "red one." Then I asked what would stand for two red beads (R_2). A child was asked to come to the board to write an answer.

I announced that I wished to add these red numbers. I wrote "$R_1 + R_1$" and asked what the answer should be (R_2). At the same time I held up two red beads, one in each hand. After a few minutes the children learned how to add red numbers. Then I switched to yellow numbers (Y_1, Y_2,...). They quickly saw that $Y_1 + Y_1 = Y_2$ for example. The crucial moment arose when I held up a red bead in my right hand and a yellow one in my left hand and asked what $R_1 + Y_1$ equaled. A variety of suggestions were offered, and several were worth exploring.

Many children show a lot of creativity in such a situation, as I have also seen in other presentations of this lesson. A frequent answer is "RY_2."

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x 1 I use the term "algebra" where many would say "arithmetic." "Algebra" is more appropriate here, since the algebraic properties of addition are the object of concern, not just rules for addition.

x 2 The following version of this lesson is appropriate for second and third grade classes. It is the version I used with my second graders.

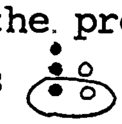
It is not usual that the answer " zero" is given, for there is no reason that it should be the answer. In this lesson after a minute or two I said that we would all agree that $R_1 + Y_1 = 0$. I held up a red and yellow bead and said: " I am holding zero." Older children might have objected that adding unlike objects is not permitted. However, they could have been persuaded to consider what the consequences would be if $R_1 + Y_1$ were to equal 0. (As a result they would be in the situation of assuming $R_1 + Y_1 = 0$, just as the other children were.) The second graders then inferred that $Y_1 + R_1 = 0$ and that in general $R_n + Y_n = Y_n + R_n = 0$, where n is any positive integer (they did not infer the equation itself, but all instances of it corresponding to definite values of n).

The physical model consisted of two stacks of beads placed on the frame. For example, the stacks for the problem $R_2 + Y_2$ were stacks of two red beads on the children's left and two yellow beads on their right.

In all four of my classes pupils were able to justify $R_2 + Y_2 = 0$ by observing that $R_1 + Y_1 = 0$ and $0 + 0 = 0$. Furthermore, they showed how to decompose the stacks into two parts which each represented $R_1 + Y_1$. The remaining case to consider was $R_m + Y_n$, where m and n were arbitrary positive integers. Thus, I asked: "What is $R_2 + Y_1$?" Many pupils discovered that it should be R_1. Moreover, they discovered for themselves that the way to see it was to hide the part of the bead stacks up to the level of the smaller stack and see what beads were left ("hide the zero").

After these first lessons the second grade had a computational technique in a physical model for adding red and yellow numbers. In both primary grades a lot of practice with the beads was needed, although the brighter children began to figure in their heads right away.

The understanding of the method represented an important discovery for them.

Two variants of the method will be described, the latter being algebraic in character. The first was introduced to the second grade, after the bead method of adding red and yellow numbers was understood. On the blackboard filled-in circles (•) represented red beads, empty circles (o) yellow beads. A model for $R_2 + Y_1$ was . Rather than "hide the zero," I suggested putting a circle around those "beads" which made zero ("circle the zero"). The children enjoyed coming to the blackboard to circle the zero. For the problem $R_3 + Y_2$ a second grader circled as follows . I agreed that he had circled zero. The essence of the discovery method is to get the class itself to explain why one doesn't obtain the answer that way. They should formulate for themselves the fact that you must circle a maximum number of beads representing zero . When they do it, they understand far better than if told by the teacher.

I had fun putting the beads in considerable disarray and having children circle the zero. Also they saw that there was more than one way to do it.

Example: 

The other variant is called the "split system" by Johntz. It is best illustrated by the following example:

$$R_3 + Y_2 = (R_1 + R_2) + Y_2 = R_1 + (R_2 + Y_2) = R_1 + 0 = R_1.$$

I found that it was quite natural for the sixth graders.

The fourth graders had learned it from Johntz the year before. I should have drilled them on it at their seats, for they needed work on it, but they knew it too well for discovery questions to be challenging. Thus, I never did much with it in the fourth grade. The second and third graders had trouble with the split system, although other discovery teachers who took the time taught it successfully to third graders. It is necessary to develop a rigid format and then to do many examples. Notice that the idea of "substitution of equals" is used at the step where R_3 is "split" into $R_1 + R_2$.

Johntz' format is:

$$\underline{R_3} + Y_2 = \square$$

$$\underline{(R_1 + R_2) + Y_2} = \square$$

$$R_1 + \underline{(R_2 + Y_2)} = \square$$

$$\underline{R_1 + \qquad 0} = \square$$

$$R_1 = \square$$ [x]

At the end one sees that R_1 goes in the last box, hence in all the preceding ones. (Each line is an open sentence equivalent to the others. These notions will be discussed later.) One might learn to do the split system just by using the format. The main trick then is to see how to split in passing from the first line to the second.

If the other algebraic ideas involved in the above derivation are avoided and the format is used in a purely formal fashion, second and third graders can do it fairly readily.

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[x] $\underline{\qquad}$ is used to show that an equal quantity has been substituted for a given quantity.

However, when one of my second graders observed "It ain't nuthin' fun," and very few had caught on,[x1] I decided to leave it. I did not pursue it with the third grade either. I believe that the correct order of presentation of the split system would be to discuss the notion of equivalent open sentences, the fact that substitution of one quantity for an equal one preserves equivalence, the associative property for binary operations, and the fact that 0 is an additive identity. (Perhaps parentheses and the associative law of addition should be avoided altogether, for since addition _is_ associative $R_1 + R_2 + Y_2$ makes perfectly good sense.) Fourth graders are ready for these ideas --- mine knew them in part through Johntz' teaching, and I continued to develop them.

After the initial intellectual discoveries concerning the addition of red and yellow numbers had been made, I introduced other topics. However, I kept coming back to the problem. A useful initial drill in all classes but my sixth grade was to fire questions around the room such as: "What is $R_{23} + Y_{27}$?"[x2] These questions keep children alert and give the slower children questions they can answer correctly. Also dittoed quizzes were useful.

Only after a class develops facility does one make the nominal change that henceforth "R_1" will be written "$^{+}1$" (positive one) or simply "1" and "Y_1" will be written "$^{-}1$" (negative one). This change caused no difficulty, and so in a very short period of time the children learned to add positive and negative numbers. (It had to be reviewed from time to time throughout the year.)

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x1 One of the second graders, Kevin, was outstanding in his ability to understand mathematical concepts. He was almost the only one who could do the split system.

x2 With second and third graders one should rarely bring in numbers above 20 in this context, for the difficulty in thinking about large numbers is extraneous to the basic addition problem.

Several children noticed that $^{+}5 + {}^{-}2$ can be computed by subtracting 2 from 5. The problem is to keep pupils from thinking of addition of a positive and negative number as actually being subtraction. Subtraction itself should be studied separately and include subtraction of signed numbers.

Whereas the second and third graders never tired of these addition problems, the fourth and sixth graders did not wish to spend much time on it. Actually the sixth graders did not fully understand it, but in view of the control problems I felt it necessary to go on to something which engaged them more (truth sets of open sentences). However, they did a little work with the split system (most could do it), and in that context I brought up the associative law of addition and the use of letter variables to stand for arbitrary numbers. Thus, several sixth graders learned to prove $(a + (b + c)) + d = (a + b) + (c + d)$, for example, by successive application of the associative law. The difficulty was that the control problem made it hard for me to do anything in depth. It would have required considerable practice for very many to learn this technique.

Also in conjunction with the split system, I asked the sixth graders how many ways a positive integer n could be written as the sum of two positive integers. After seeing several examples, most discovered that the answer was $n - 1$ <u>for particular values of n.</u> However, when I introduced the letter "n" and asked them to view it as standing for an arbitrary positive integer and then to give me the correct answer in the general case (in terms of n), no one could do it. Most suggested m as the answer, because m comes just before n in the alphabet. It is a genuine insight to understand what is meant by "$n - 1$" where n stands for a number, and they were not ready for it. I asked them to consider the problem of adding an initial segment of the positive integers.

Thus:

$$1 + 2 + 3 + 4 = \frac{4\ (4 + 1)}{2}$$

Some could see that:

$$1 + 2 + \ldots + n = \frac{n\ (n + 1)}{2}$$

is a generalization of such facts, and they gained some idea of the use of a letter variable. However, the understanding of variables did not come until much later when the notion was introduced in two very intuitive ways.

The next major topic, which actually was introduced on about the third or fourth day, covered the notions of "open sentence," "truth set of an open sentence," "false set of an open sentence," "true," and "false." Many people restrict the notion of open sentence to equations and inequalities, but I feel it should be used in full generality. Informally one might say that an open sentence is a sentence with at least one empty box ($\square$) in it.[x] This definition is not very good, since an open sentence is an incomplete sentence, i.e., not really a sentence. A correct definition is rather involved. Usually it can be taken to be an expression with one or more empty boxes such that if a number is placed in $\square$ at each of its occurrences, a sentence is obtained. For children it may be better simply to use open sentences without defining what they are.

In any event, the problem of adding red and yellow numbers should be quickly rephrased. For example, one might ask "What should I put in the box?" where

$$R_3 + Y_5 = \square$$

appears on the blackboard. I did this and then announced that I was going to call "$R_3 + Y_5 = \square$" an open sentence and explained that an open sentence had to have one or more empty boxes in it. The question can also be rephrased: "Which (red or yellow) numbers make the open sentence true, if you put them in the box, and which ones make it false?" We agreed to put the

x Initially $\square$ was the only variable.

numbers which make it true between braces to the right of the open sentence (using commas to separate numbers, if more than one number works), and to call the result the truth set of the open sentence. Thus, addition problems reduced to finding truth sets of open sentences. A closed sentence is a sentence with no empty boxes. Thus after placing a number in the boxes of an open sentence, a closed sentence would also be a sentence with no boxes. Closed sentences are either true or false (but not both). A question which helps to clarify this fact for the children is the following: "Is $5 \neq 4$ true or false?" Trying to understand the relation between truth, falsehood and negation is a stimulating challenge.

Most children call a number which makes an open sentence true, if one puts it in the boxes, "the answer." They should be steered away from this usage. One way is to introduce the false set, the set of all numbers which make an open sentence false. Moreover, it is necessary to use open sentences with more than one element in the truth set. Example: $^{+}5 \neq \square$. If the children know that open sentences can use = or other verbs, they can be asked to give an open sentence with more than one member in its truth set. The discovery method thrives on such questions.

I asked my third grade whether open sentences were true or false. After they saw that both true and false sentences could be obtained by placing different numbers in the boxes, they became puzzled. The prevailing answer was: "It could be either." Of course, I wanted someone to say: "It's neither," but no one obliged. The trick in successful use of the discovery method is to find leading questions such that the children will discover the correct answer to the original question. Thus, I asked them to consider the open sentence:

$\square$ sits in the third row.

They saw that the truth set was the set of children sitting in the third row: $\{$Michael, Freda, ...$\}$.

Then I asked if the open sentence was true or false. They all answered to the effect that it was true if one put "Michael," "Freda," ... in the box. I asked: "Is it true or false as it stands?" Essentially the children repeated the original answer and refused to consider the open sentence "as it stands." I had to give up on that point. I gave the answer (contrary to standard practice in discovery teaching). My reward was what I deserved --- they did not accept it.

The second graders were the most intellectual class. They enjoyed debating just such abstract questions as whether an open sentence can be said to be true or false. Whereas they did not resolve it fully, they understood the distinction between an open sentence and the result of putting numbers in its boxes. An example of the enthusiasm with which fairly abstract questions were discussed by this class is provided by the following open sentence:

$$^{+}3 < \square \quad \text{and} \quad \square < {}^{+}7$$

Order notions had been introduced, and I was attempting to clarify the notion of an open sentence such as:

$$^{+}3 < \square < {}^{+}7$$

The children realized that two open sentences connected by "and" form a single (new) open sentence (an insight which sometimes escapes logic students in college). I asked what numbers belonged in the truth set. Someone suggested $^{+}8$. I asked what I should do with $^{+}8$, and they all chimed in: "Put it in both boxes!" (the same number must always go in all boxes). Then I asked if the resulting sentence was true or false. A variety of answers were eagerly hurled back at me: "The first part is true, and the second part is false!". "It's true and it's false!" I warned them that a closed sentence must be either true or false, but not both. Perhaps the greatest virtue of the class was that they

would not accept such statements on faith. As a slight gesture of deference to me, they decided that it was "true-false" (some sort of hybrid truth value). The nature of this solution was that it can be consistently applied to other conjunctions (possibly with several occurrences of "and"), whenever the truth values of the conjuncts are known. The children consistently applied this idea,[x1] so that we found ourselves considering sentences which are true-false-true, for example. By bringing up several everyday sentences with "and" occurring, I was able to bring them to see that the original open sentence had $^{+}8$ in its false set. However, they were still reluctant to give up the system they had developed.

Not all the time could be used for new discoveries. The second graders needed a great deal of practice with open sentences of the form: $a + b = \square$ (a, b integers). They never seemed to tire of it. For the first three months they had at least a little drill with such problems every day, although I varied the approach. One day I would fire questions all around thr room: "What's $^{+}5 + {}^{-}7$?" etc. Whenever a child got stuck or gave the wrong answer, I wrote on the board $^{+}5 + {}^{-}7 = \square$ and had him come to the board, draw a bead representation for the problem, circle the zero, and write the truth set next to the open sentence.[x 2] Thus he eventually got the correct answer. Another day I had them practice with the real beads.

To stress the notion of an open sentence itself, I frequently set up two stacks of beads and asked someone to put up the open sentence which "goes with" the stack. Sometimes I wrote: $\{^{+}3\}$, for example, and asked who would like to come up and write an open sentence with that truth set.

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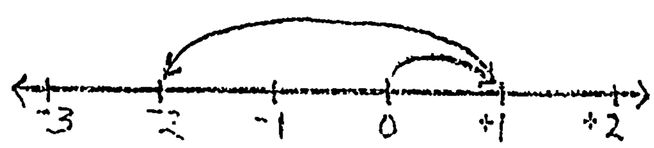
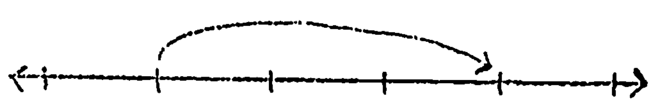
x 1 They were always quite consequential --- on April 1 I told them that a mathematician at the University of California had just made the discovery that $1 + 1 = 3$. The response to my April Fool joke was that one boy replied: "That means $1 + 2 = 4$, $1 + 3 = 5$,..."

x 2 In this class the children loved to come to the blackboard to put on a good show for their classmates.

Another method of computing the solution of equations of the form $a + b = \square$ is to count on a number line (or count rungs on a ladder.) Some point (or rung) is assigned to 0 and the integers near 0. The second graders learned this technique to some degree, but the third grade class developed it further. They first studied sentences of the form $a + b = \square$ in much the same fashion as the second graders. A natural way to use a number line to find the truth set of an open sentence such as $^{+}1 + {}^{-}3 = \square$ is to start at 0, count one to the right and then three to the left:



First of all, the children had to learn to count a given number to the right or left, so naturally the difficulty arose whether to include the starting point in the counting. One might count four to the right as follows:



Many children wanted to count that way. To counter the impulse, I suggested that the marks on the line could stand for cities one mile apart from the neighboring ones and that motion to the right be eastward motion, etc. Then I asked: " If you start at city 0 and travel one mile east, what city are you in? What if you travel two miles east?" I varied the starting point: " What if you travel three miles west starting at city 1?" The correct mode of counting was discovered rather gradually, but finally I was able to return to the open sentence $^{+}1 + {}^{-}3 = \square$. Using the notion of distance in conjunction with the notion of counting removed the difficulty.

In order that all children could master the technique, I formalized it slightly by introducing the format:

START AT ZERO.

GO ____________.

THEN GO _________.

END UP AT $\square$.

In finding the truth set of $^{+}1 + {}^{-}3 = \square$, first fill in the blanks with "Right 1" and "Left 3," respectively. Then use the first three lines as instructions for counting on the number line (and drawing in the arrows). At the third line the word "THEN" reminds the children to begin counting left 3 at the stopping point of the previous count (rather than go back to 0). Finally the student looks at his stopping point and enters it in the box.

Aside from providing another technique for adding integers, the above method helps introduce the number line and the ordering of the integers. We defer discussion of this topic to a later point.

The second and third graders handled more involved open sentences. It was always fun to look for the truth set of a sentence like:

$$^{+}5 + {}^{-}3 \neq \square.$$

After two or three values in the truth set were supplied, some pupil would point out impatiently that every integer except $^{+}2$ belonged in the truth set and that the false set was $\{^{+}2\}$.

Although I gave the second graders some experience with open sentences of the form

$$a + \square = b$$

$$\text{and } \square + \square = a$$

the third graders worked extensively with such open sentences and even more involved ones.

First of all, it was necessary to emphasize that the same number must be put in each box, if there is more than one. Even sixth graders were tempted to solve open sentences with several boxes as follows

(for example):

$$\boxed{+1} + \boxed{+3} = {}^{+}4 \cdot \{+1,+3\}.$$

The fourth graders avoided this mistake, for Johntz had schooled them well during the third grade. Once this restriction was clear, the third graders were ready for a fairly large class of open sentences involving addition, constants and one or more boxes. The first type to consider is exemplified by one like ${}^{+}5 + \square = {}^{+}3$. The nature of the problem excited the children, for they realized that it was necessary to guess an answer and test it, rather than directly add. Thus, if a child suggested ${}^{+}8$, he saw immediately that it didn't work, once it was placed in the box. Until familiarity with problems of this type developed, considerable eager, competitive guessing went on. Fairly soon many saw the correct way to think about the problem. Their sense of pride was obvious.

Successively sentences such as:

(1) $\square + \square = a$

(2) $a + \square + \square = b$

(3) $a + \square + \square = b + \square$

(4) $\square + \square + \square = a$

(5) $a + \square + \square + \square = b$ (a,b integers)

were introduced. The teaching problem was to provide more and more of these increasingly difficult problems for the children who were getting the idea quickly, while also giving problems of the type

$$a + \square = b \quad x$$

to those who still hadn't discovered the idea.

The third graders did not reach a point where they could handle open sentences of types (2), (3) or (5) systematically. However, they did develop the notion that if a candidate for the truth set of such an open sentence made

x The idea involved is basic to subtraction and may be developed more formally when subtraction is taught. The children preferred an intuitive approach.

the right side "almost" equal to the left side (upon substitution in the boxes), then that candidate must be almost equal to a member of the truth set.

The fourth graders and sixth graders learned to solve problems of this sort in a fairly systematic fashion, although I only gave them a technique long after they had developed their own.

Some of the development will be sketched. The fourth graders, not knowing division at the time, were able to visualize solutions for

$$\underbrace{\square + \square + \dots + \square}_{n \text{ times}} = a, \quad ^{x1}$$

provided that a was an integer divisible by n (a could be arbitrary in case n = 2, 3 or 4). The sixth graders saw this much. However, several also saw that a/n was the solution.

The interesting cases were the ones of the form

(6) $a + m\square = b + n\square$ [x2]

Both the fourth graders and the sixth graders realized that the difference between m and n was critical. Several fourth graders quickly reduced the problem for m = n + 1 to solving

$$a + \square = b$$

They had to wrestle with $a + \square + \square = b$.

The discovery made by one boy to solve the problem $^{+}10 + \square + \square = {}^{-}4$ is interesting. After thinking awhile he said: " Draw one box with a line down the middle (instead of two boxes). Then $^{-}14$ goes in the box, so $^{-}7$ goes in one half and $^{-}7$ in the other." On the basis of this idea, he concluded that $^{-}7$ was the solution --- just pull the halves apart.

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x 1 The notation $n\square$ was introduced as an <u>abbreviation</u> for the left side.

x 2 It should be emphasized that the letters a,b,m and n did not occur in the actual instances of this open sentence considered by the children.

The rest of the class understood him, so they all had fun solving several more problems of the same type. After that, problems of type (6) with $m = n + 2$ could be reduced to the foregoing type.

As the difference between m and n increased, some difficulty was encountered. The children reverted to guesswork frequently. The fourth graders had difficulty with the sentence $\square + \square + \square + {}^{-}10 + \square = \square$, although several pupils finally solved it. In testing a candidate for the truth set, they had trouble evaluating the left side. I distributed dittoed problem sheets with a large variety of problems of types (1) - (6). The more interested pupils worked on them very eagerly, even at home. In some cases I let several constants occur on either side of the equality and interspersed them with the boxes (as in the problem just-mentioned). For both the fourth and sixth graders these problem sheets were a great challenge; they evoked some of the most enthusiastic responses of the year. In the open sentences for the sixth graders I let $\square$ have positive coefficients, so that, for example, "$(5)(\square)$" was an expression for multiplication of 5 and the variable "$\square$." I had to avoid equations with negative solutions, since multiplication of signed numbers had yet to be discussed. Several pupils taught themselves to collect terms, reducing a given problem to one of type (6), which then reduced to

$$a + (m - n)\square = b$$
$$\text{or} \quad a = (n - m)\square + b$$

Rather than bring both constant terms to one side of the equality, they solved

$$a + \square = b$$
$$\text{or} \quad a = \square + b,$$

and then divided by $m - n$ or $n - m$, respectively. I never did explain to them the explicit techniques developed in high school algebra for such problems, but I am con-

vinced that they will need almost no explanation when presented with arbitrary linear equations in one unknown in high school.

However, at a later date we discussed the notion that two open sentences are equivalent exactly when they have the same truth set. Rather than stress the definition of equivalence, I applied it, using the arrow notation already mentioned in the discussion of the split system. The sixth graders became adept at transforming one equation into an equivalent one. Thus, they solved $(\square + {}^{+}4) + {}^{-}3 = 0$ as follows:

$$\underline{(\square + {}^{+}4) + {}^{-}3} = 0 \qquad \{{}^{-}1\}$$
$$\downarrow$$
$$\square + \underline{({}^{+}4 + {}^{-}3)} = 0 \qquad \{\quad\}$$
$$\downarrow$$
$$\square + \quad {}^{+}1 \quad = 0 \qquad \{{}^{-}1\}$$

The notions involved include certain laws of algebra (which they learned) and the fact that substitution of equals preserves truth sets. The latter fact was made explicit, but not stressed. The actual reason it was not stressed and that we did not drill on the method is that the discussion came at a time when I was having a control problem with the sixth graders. The pressure forced me to stress activity and solution of explicit open sentences (where they tended to compete for the right answer), rather than intellectual discussion, during which they became resistant. Also drillwork on concepts only partially learned was hard to do. As a result, about half the class had a good understanding of what they were doing. About the same percentage really understood the basic ideas just discussed for solving linear equations. In fact, there were about three mental dropouts whom I never reached all year. By way of anticipation of the conclusions, where the sixth grade is concerned, I would say that I had an outstanding " half-class" which did some high-level mathematics, while the other half was in varying degrees mentally absent. I blame the

control problem mostly. Moreover, I have the feeling that sixth graders in a school where they are the upper grade begin to take their old school not so seriously and to think about next year (and the new social life). At Lincoln all the sixth graders were difficult to handle. They knew their way around the school and had a variety of tricks for getting out of class. They were sometimes defiant of authority and, in any case, resistant. The latter circumstances seem quite natural to me, but the sixth grade teacher should be strong. The teacher of my sixth graders made a good effort and kept the problem within bounds, although she did not master it.

With the fourth grade I did stress the notion of equivalent open sentences. Also two closed sentences were defined to be equivalent whenever they both have the truth value (true or false). These notions were used to find the truth set of open sentences such as

$$^{-}3 + \square + {}^{+}2 + \square + \square + \square + \square = \square + \square + {}^{+}11,$$

or to check the truth of closed sentences such as:

$$^{-}7 + ({}^{+}8 + {}^{-}20) = {}^{-}29 + {}^{+}10$$

In the former case I would ask: "What is an open sentence equivalent to this one where all the boxes are on one side of the equal sign?" To answer the question, they had to realize that if one box is simultaneously removed from both sides, then the resulting open sentence is equivalent. They grasped this idea after practice with very simple open sentences. They were asked to explicitly check that $\square + \square + {}^{+}5 = \square + {}^{+}2$ is equivalent to $\square + {}^{+}5 = {}^{+}2$ and to do other similar examples. Thus the answer $^{-}3 + \square + {}^{+}2 + \square + \square = {}^{+}11$ came readily to the question above. The next question was: "What is an equivalent open sentence with all the boxes together?" They replied:

$$^{-}3 + {}^{+}2 + (3)(\square) = {}^{+}11$$

(Recall that "$(3)(\square)$" was an abbreviation for "$\square + \square + \square$"

in the fourth grade.) Then they obtained

$$^{-}1 + (3)(\square) = {}^{+}11$$

as the next equivalent open sentence. (I did not discuss substitution of equals with them. This step would have been better understood had I developed the topic separately before taking up solution of open sentences using the method of finding equivalent open sentences.) At this point I let them get the solution $^{+}4$ by intuitive means. There was good reason for not pressing them systematically to obtain the next sentence:

$$(3)(\square) = {}^{+}12$$

The reason is that they did not discover it for themselves. By continuing to solve $^{-}1 + (3)(\square) = {}^{+}11$ using an intuitive approach, they would have discovered that " you add $^{+}1$ to $^{+}11$ and divide by $^{+}3$." As it was, other subjects came up and they did not spend all the time needed to complete the method. Since I tended to structure their thinking quite a bit in teaching the part of the method which yielded $^{-}1 + (3)(\square) = {}^{+}11$ (for example, I suggested that cancellation of boxes preserves truth sets and then got them to verify it in particular cases), and since the development extended over about three weeks, they began to tire of the matter. In order to keep the mathematics fun, I had to allow the children to just play with it from some point on. In any case, the best pupils were able to solve $^{-}1 + (3)(\square) = {}^{+}11$ readily, and all of them were able to reduce sentences such as

$$(53)(\square) + {}^{+}4 = (51)(\square) + {}^{-}2$$

to $$(2)(\square) + {}^{+}4 = {}^{-}2$$

This achievement put them well on the way to understanding what really underlies the methods for solving equations developed in high school algebra. I believe that with one more year of discovery teaching in which the techniques for obtaining truth sets of open sentences by finding equivalent ones are covered, these children will know more about

algebra than tenth graders do.

As already mentioned, my sixth graders did learn these techniques explicitly, but about half learned the methods implicitly by discovering them for themselves. My contribution was some brief coverage of the method (one day) and automatic use of the substitution lines and arrows. Thus rather than point out that the following transformation "preserves truth sets"

$$\underline{(^{-}3 + \square)} + {}^{+}10 = 0 \quad \{\ \}$$

$$(\square + {}^{-}3) + {}^{+}10 = 0 \quad \{\ \},$$

I used language such as: "$^{-}3 + \square$ is the same as $\square + {}^{-}3$, so I can replace the one by the other." The emphasis was on the activity of transforming, rather than on formal justifications.

The sixth graders tried a few quadratic equations, mainly because they suggested them. Sometimes I would ask: "Give me an open sentence." Whereas I expected some reasonable equation, they would always strain and come up with something complicated. Thus we were led to consider

$$\square + \square + \square - (\square \times \square) = {}^{+}20,$$

a quadratic equation with no real roots. In this case, they were able to conclude that the truth set is the empty set (after considerable guessing). In general I avoided quadratic equations on the grounds that the class could easily afford to spend a full year on much simpler material.

Before concluding the discussion of open sentences in one unknown, I wish to include a further example of the work of the third graders. I wrote on the board:

$$+ \square = 0$$

and said that I had written a number to the left of the plus sign, but that it was too small to see. (In fact, I made a minute scribble.) Then I said that this number had another name: Q W I G. I asked for the truth set of the open sentence. There was great puzzlement.

Many children tried to guess what Q W I G was, so I said that I had forgotten, but that they could still answer the question. Finally one of the brightest girls said: "$^{-}$QWIG." Everybody agreed that QWIG + $^{-}$QWIG = 0. A mathematical difficulty lurks in this situation. What if QWIG stands for a negative number? No mention had been made of an expression such as $^{--}5$. I wished to introduce the notion of letter variables (as well as boxes) to the third grade. At the same time I wanted a single equation which would express the law that a number added to its additive inverse always yields 0. Thus, I had to introduce a unary operation symbol standing for the additive inverse operation.

I did introduce a symbol for this purpose: "$^{\circ}$". As a result (for example),

$$^{\circ +}5 = {}^{-}5$$

$$^{\circ -}5 = {}^{+}5.$$

The first equation is read: "The opposite of $^{+}5$ is $^{-}5$." The class learned very quickly what was intended. They figured that 0 is its own opposite. Then it was possible to consider open sentences such as

$$^{\circ}\square = {}^{+}10$$

$$^{\circ}\square = 0$$

$$^{+}7 + {}^{\circ +}7 = \square$$

$$^{\circ +}2 + {}^{+}3 = \square$$

$$^{-}1 + {}^{\circ +}1 = \square$$

and $\square + {}^{\circ}\square = 0.$

Moreover, the mathematical difficulty with the "QWIG" problem was eliminated. The truth set of QWIG + $\square$ = 0 contains $^{\circ}$QWIG. The third graders did not like this answer as well as $^{-}$QWIG. Also they had trouble with $\square + {}^{\circ}\square = 0$.

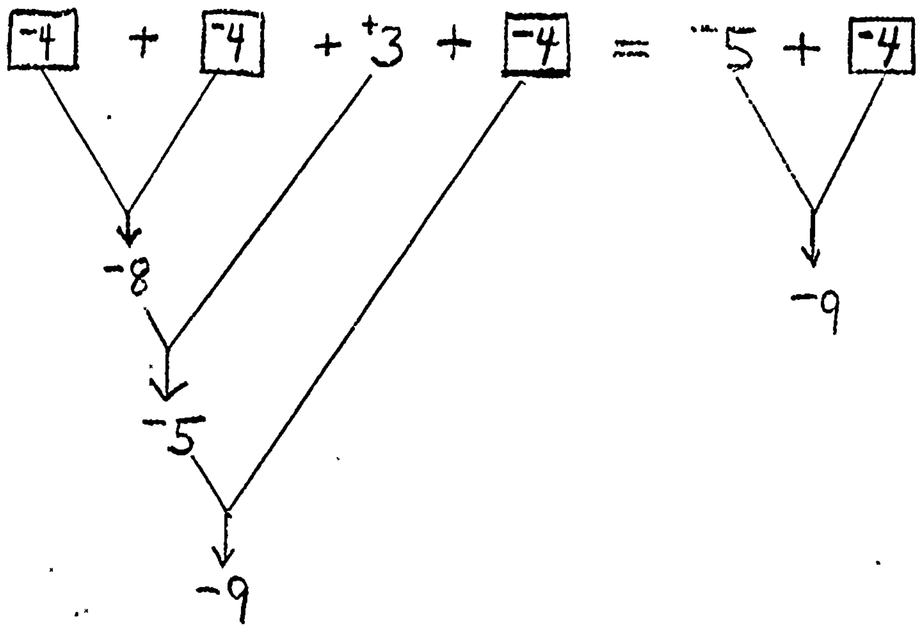
They finally realized that the truth set contains "all numbers," but I do not feel that the goal of having them discover and formulate the fact that a number plus its additive inverse is always 0 was reached. They seemed unconvinced and not quite sure what the point of it all was. As a result, I favor using " $^-$ " as a symbol for the additive inverse operation, since this usage is consistent with prior usage of the symbol, (providing we agree, for example, that $^+5 = 5$). In introducing negative numbers one should stress that this symbol is different from the subtraction symbol, which is written lower on the line. If " $^-$ " is used instead of " ^{0} " it seems clearer that the equation $\square + {}^-\square = 0$ contains all numbers in its truth set (put 5 in both boxes --- the result "$5 + {}^-5 = 0$" is close to the original fact "$^+5 + {}^-5=0$," which the children learned the first week).

I actually did let " $^-$ " be the additive inverse symbol in the sixth grade class. I also let "$^+$" stand for the identity mapping on the reals. One of the problems solved by the sixth graders was: Find a rule for solving problems such as

$$-+---++-+--+-3 = \square .$$

Two methods were developed. The first amounted to counting the number of negative signs. The other required crossing out all $^+$'s and then crossing out the $^-$'s two at a time, until one or none remained.

In addition to the above difficulties with " ^{0} ", the children also had trouble checking the truth of equations such as: $^{0-}4 + {}^-4 = 0$. They first should have learned that the trick is to compute $^{0-}4$ separately and substitute the result. Thus, substitution of equals should have been studied separately (as a transformation on sentences which preserves truth sets and truth values). Experience with both the third and fourth graders led me to this conclusion. In both these grades I developed a format for checking the truth of complicated closed sentences, as seen in the following example:



This method was needed (in both the third and fourth grades) for efficiently checking whether a number belongs in the truth set of such open sentences.

A topic closely related to that of open sentences in one unknown, but conceptually more involved is the topic: open sentences with several unknowns. It was introduced for two (and sometimes three) variables: □ , ◯ and △. The latter two were called "egg" and "triangle." The second graders did not study such problems. One of the main ideas is that the variables are in a definite order --- □ is the first and ◯ is the second.(I do not speak of their order of occurrence in a given open sentence). The truth set of an open sentence in which □ and ◯ occur does not contain numbers. It contains ordered pairs of numbers. Thus, if p(□,◯) is an open sentence with one or more occurrences of both □ and ◯, then the truth set of p(□,◯) is taken to be the set of ordered pairs (x,y) such that x and y are numbers for which p($\boxed{x}$,$\boxed{y}$) is true. The sentence p($\boxed{x}$,(y)) is the result of putting x in each occurrence of □ and y in each occurrence of ◯ in p(□,◯), x and y being particular numbers. Thus, the first variable corresponds to the first co-ordinates of the ordered pairs in the truth set, the second variable to the second co-ordinate.

Children can learn to find truth sets of such open sentences by studying examples. The best way to get across the correspondence of variables with co-ordinates is to simultaneously

study two open sentences such as:

$$\square + \square = \bigcirc$$

$$\text{and} \quad \bigcirc = \square + \square \ .$$

It is easily determined by even third graders that the truth set of the first open sentence contains pairs such as $(^+1, {}^+2)$, $(^+2, {}^+4)$, . . . Although I told the third graders that to test an ordered pair, they had to put the first co-ordinate in the boxes, the second in the eggs, they actually discovered it for themselves by first getting the wrong truth set for the second open sentence. They accepted pairs such as $(^+2, {}^+1)$, $(^+4, {}^+2)$, . . . Since they had learned that $(^+1, {}^+2)$ and $(^+2, {}^+1)$ are distinct, they realized something was wrong, when I pointed out that the two open sentences had the same expressions equated, only in different orders. I urged upon them that the truth sets should be the same (they agreed) and asked what was wrong. They realized that in testing $(^+2, {}^+1)$ for the truth set of $\bigcirc = \square + \square$, they should have put $^+1$ in the egg and $^+2$ in both boxes. The conclusion would have been that $(^+2, {}^+1)$ goes in the false set. After this example they did not forget the correct rule very often.

For the third and fourth graders, on Johntz' suggestion, I purchased peg boards (4'x4'), sprayed them with white paint, and used golf tees (as pegs) to set up two co-ordinate axes (the $\square$-co-ordinate and the $\bigcirc$-co-ordinate). Thus, they were able to graph the truth sets of open sentences such as

$$\bigcirc = \square + \square \ .$$

They had to develop for themselves rules prescribing where, for example, to place a peg for the pair $(^+3, {}^+6)$. Many wished to place the peg at either $(0, {}^+9)$ or $(^+9, 0)$. With some difficulty the third graders saw that the same idea would require placing the peg for the pair $(^+2, {}^+7)$ in the same place. I said that no two ordered pairs should be represented by the same peg.

The fourth graders already knew something about it, so they had little difficulty in deciding on the correct procedure.

The third graders had to be helped. They found it hard to count, since there were no ruled lines (horizontal and vertical), although pegs were placed in every fifth hole on the co-ordinate axes and at the origin, so as to determine the axes. As a result, inessential mechanical difficulties stood in the way of their discovering the rule. When I suggested the rule, the class learned it correctly. Then I tried using graph paper or ruled lines on the blackboard. They could visualize much better where to plot a point for a pair.[x] Also I had them write the pair next to the point plotted. That way I could check their work when they were plotting several points. I am sure that it helped them to learn to "identify" points with pairs.

We then plotted the truth sets of various simple open sentences such as

$$\bigcirc = \square + \square$$

$$\bigcirc = \square + \square + \square$$

$$\bigcirc + \square + \square = 0$$

$$\bigcirc = \square + {}^{+}1 .$$

Although it took them some time to write down enough ordered pairs, they would finally get several points plotted. A few students were sufficiently accurate that they could begin to see that the points were all in a straight line.

The fourth graders proceeded very quickly. They saw immediately that such equations have straight line graphs. I began asking them to tell me where a line would go without first finding any of the ordered pairs in the truth set of the equation. The better pupils were well on their way to making a general analysis of the graphs of linear equations. They could distinguish between lines of positive and negative slope and relate the distinction to the form of the equations. The two most articulate fourth graders could formulate the distinc-

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x I believe the peg board would be successful either if the children were older (as the fourth graders were), or if the visual image were simplified a bit. Many children in my very large third grade sat far from the front of the room.

tion in terms of whether all variables occurred on the same side of the equation or whether all boxes occurred on one side and all eggs on the other. Many others understood this basic idea. They knew that the constant terms in an equation determined whether its graph passed through the origin. They were beginning to see what determined flatness and steepness of a line. Curiously enough, most pupils felt that the graph of $\bigcirc = \square$ was a steep line. I tried to suggest that it was just between being flat and steep.

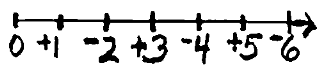
In the sixth grade I did a little work plotting points for ordered pairs (on the blackboard), but did not discuss graphing equations. They immediately learned the rule for plotting ordered pairs. We then played tic-tac-toe, using points with co-ordinates between ${}^{-}5$ and ${}^{+}5$.

With each class I played a variety of mathematical games. The children loved to compete with each other. Several games were so popular that pupils would come up to me during recess and implore me to play them in their class. One of the games was number-guessing --- the class had to guess a number using a minimum of yes or no questions. Another was of the type where a fixed number of circles are drawn on the board. Two people take turns erasing circles (one, two or three per turn, for example), until all are erased. The winner is the one who erases the last circle. The sixth graders learned quickly the general strategy for all variants of this game.

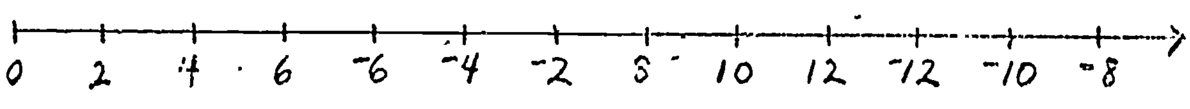
The next level of mathematical complexity is attained when the ordering of the number system is introduced. The upper graders already knew the correct ordering, so I did not discuss it extensively with them. The second and third graders, especially the former, became engrossed in some interesting problems. Of course, I wanted everyone to become familiar with $<$, $\leq$, $>$ and $\geq$, so that they could find truth sets of open sentences involving these relations. The truth sets themselves became more interesting (linear equations only had truth sets with no numbers, one number or all numbers). Both classes learned to do many problems such as $\square < {}^{+}1$, $\square \leq \square$, $\square < \square$, and ${}^{-}3 < \square \leq {}^{+}2$, to cite a few examples. It took a while before they refrained

from confusing $<$ with $>$. It also took a while to understand the difference between $<$ and $\leq$. A valuable exercise consisted in putting a set on the blackboard, say $\{{}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3\}$, and asking for an open sentence having that set as its truth set.

However, more fruitful than the (rather formal) study of truth sets of inequalities was the study of number lines, the usual physical models for linear orderings. First I will describe my experiences with the third grade. They were quite familiar with the standard number line for the non-negative integers. Since they had never heard of negative numbers before I introduced them, they had no inkling what sort of number line to use for all the integers. I simply asked them to give me a number line which would have on it the positive numbers, zero and the negative numbers. The positive numbers were required to be in the same order as before. A lot of imagination was put into some of the answers! One boy suggested:

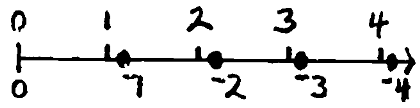


I asked: "Can anyone tell me a number not on this number line?" One reasonable answer to the question is that $^{+}7$ is not on it. We had discussed the meaning of the arrowhead. They knew that this line was supposed to go on "forever." They had rather shakily accepted the notion that if an arrowhead were placed on the right hand tip of their old number line (which ran from 0 to 100), then that would mean that all the numbers after 100 are "really" on it. Thus, they could see that $^{+}7$ would have been written on the above line, had I put a few more numbers down before attaching the arrowhead. They realized that $^{+}7$ belonged on this number line. Then someone realized that $^{+}2$ was missing and that it would never be placed on the line, if we continued the pattern. This very problem led me to devise jumbled orders in which a pattern existed that was to be periodically repeated. They learned to continue these patterns, so that they could even predict whether a given number would occur on the corresponding number line. For example, they could continue the pattern illustrated on the following number line:



and they knew that 21 and $^{-}21$ would not be on it.

When I pursued my request for a number line containing all the integers (with the positive numbers in their usual order), they came up with the following example:



I had also insisted that no negative number could be between two positive numbers. I consider the girl who drew the above number line on the board to be quite creative (it was indicated by her responses all semester).[x] In order to satisfy the requirement that no negative number could be between two positives, she wrote the names for the positive integer points above the line and the names for the negative integer points below the line. She firmly maintained that no negative number was between two positives. Her certainty was by itself something valuable. She was not easily persuaded to give up one of her own notions. Only a few of her classmates possessed comparable intellectual self-reliance, although it seemed to abound in my second grade.

The issue raised by her answer is very complex. I explained that one writes the name for a number next to the point on the line where the number itself really goes. Although a mathematician might prefer an explanation such as: "One writes _a_ name for the number next to the point on the line _corresponding_ to the number," it is an explanation which would just confuse the issue for the children. The distinction between names and the objects they denote, as well as the distinction between points on a number line and numbers themselves, can be made with children, but it seems to require considerable care and initially should not be treated in the midst of some other discussion. Later on I did some work with all my classes on the name-denotation problem, for it is really the key to a proper understanding of "substitution of equals" in algebra.

In any case, the children realized that the above number line still had negative numbers between positives, but they were

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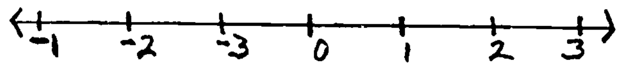
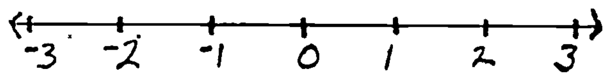
x This girl went to another Berkeley school the following semester as part of a plan to bus some children to integrated or nearly all white schools. From my point of view her departure was a significant loss.

becoming frustrated, because the original question went unanswered. I had never suggested that the number line could be doubly infinite. Accordingly, it never did occur to them. Finally I drew on the blackboard the following line: $\longleftrightarrow$.
At that point many hands went up. The first try was



I asked where $^{-}4$, $^{-}5$, $^{-}6$, and so on, belonged. The class realized that they should come after $^{-}3$, but that there was no room. The issue became clouded when several suggested that these numbers be placed to the left of $^{-}1$. After all of this discussion, which actually took several days and included part of the work with repeating patterns, someone suggested:

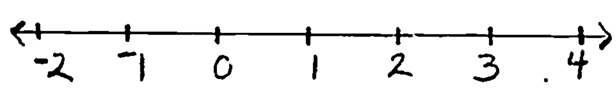


the number line for the standard linear ordering of the integers. Curiously enough, it seemed so natural to the class that they did not react as though they had just made a discovery. A discovery teacher likes to see the response of great surprise and exultation, but a great many discoveries take place quietly, gradually and perhaps (some of the time) unconsciously. In any event, once the children saw the standard number line, they forgot about all other possibilities and accepted it thoroughly. I used it to define the relation $<$: given two numbers, the number on the left is smaller than the number on the right. The consequence $^{-}5 < {}^{-}4$ seemed initially strange to the class. To put them on guard, I asked them for the smallest number they could think of. Many gave the answer zero. Then I asked for a number less than zero, and someone realized that $^{-}1$ is less than zero. Before we stopped for the day, they were down to negative ten trillion!

At this point in the development with the third graders we took up the study of open sentences involving inequalities, to which reference has already been made, so I turn now to the work of the second grade with number lines, which was rather different. The second graders were much more inventive. They

would debate abstract questions with the whole class hanging on every word. Such intense concentration was not achieved in the third grade. In order to stress the level of concentration and involvement accepted as routine by a discovery mathematics teacher, I should point out that a great many visitors observed my third grade class --- teachers, parents and administrators --- and every visitor stated that he was highly impressed with both the attentiveness and level of work. Nevertheless, the second graders were markedly better.

In the second grade I told the class at the outset how all the integers could be put on a number line. I used both horizontal and vertical number lines, so at the beginning of each hour they all had fun guessing which way I would draw the day's number line. As mentioned before, two main levels of ability were almost equally represented in the second grade. A favorite technique for involving the weaker children was to put a number line on the board having the marks for the integer points, but with the numerals missing. I would first ask for someone to place zero on the number line. Then various others would write in the other numerals, until all the marks had been used up. It was a great privilege to be chosen to place a number. If one person made an error, the next would triumphantly correct it. Sometimes the result would be a number line such as the following:



Several pupils would get excited and say (with characteristic certainty) that it was wrong. Although we had discussed the arrowheads, so that they knew that all integers were intended to be on the line, they would not accept the above line, until the left arrowhead had been erased, the line extended, the numbers $^{-}3$ and $^{-}4$ written in, and the arrowhead put back. An alternate solution was to chop off 3 and 4 on the right. In the future I was always careful initially to place an odd number of marks on the line (otherwise they would have said that I had made a mistake).

One day the brightest boy of the class, Kevin, who might well end up being a professional mathematician, if he continues

at the same relative level of performance he established in his second grade mathematics class, decided that he was going to use the following number line in the future: 2 1 0 $^-1$ $^-2$.

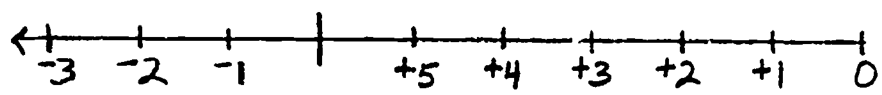
At first he said mine was wrong, but then he admitted that both could be used. He insisted that his was better. We had already defined <, so he knew that the definition would imply $1 < {}^-1$, if his number line were used. Unfortunately I do not have record of his reason, but he had a definite reason for deciding that $^-1$ should be greater than 1. Moreover, he realized that one could choose it to be either way. He simply felt that his way was better. The only sensible answer is that he was quite right --- either way would work[x] --- and the only reason for sticking with my way was that everybody else did it that way, and there might be confusion otherwise. In fact, I told Kevin that he could do it his way, as long as he realized how others make their number lines. Moreover, in discussing number lines and inequalities with other people, he should make sure that everyone knows which system is being discussed.

Such situations occurred frequently in the second grade, so frequently that the class practically learned to parrot back to me: "Either way is all right, but everybody else does it this way, so if you talk to other people, you have to do it their way, or they won't know what you're talking about." They seemed to understand and appreciate the fact that they were capable of coming up with non-standard, mathematically sound notions. They never seemed to slip into the practice of looking for the answers that would please me. Perhaps they realized that I was pleased, so that they were free to consider the mathematical questions on their merits.

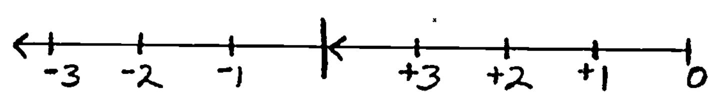
One day when we were discussing inequalities (they studied inequalities much as the third graders did), Leslie, another bright second grader, announced that she wanted to put a number line on the blackboard: $^-3$ $^-2$ $^-1$ | $^+3$ $^+2$ $^+1$ 0

x When multiplication is introduced, the algebraic structure would be affected by the choice of number lines, but the class did not know multiplication.

I asked if all the numbers were meant to be on it, or just the ones written down. She intended all the numbers to be on it. I asked the class what could be done to show where $^{-}4$, $^{-}5$, and so on, belonged. They decided that an arrowhead should be attached on the left. Then I asked Leslie if that was what she had intended. She nodded. I asked where $^{+}4$ went. She said: "After $^{+}3$." She meant to the left of $^{+}3$. Moreover, it came out that she knew she couldn't write all the numbers, but if one wanted to put a few more positive numbers on the line, one could extend the line to the right, shift the numbers on the right to the end, and then put some more on the line, say $^{+}4$ and $^{+}5$, as follows:

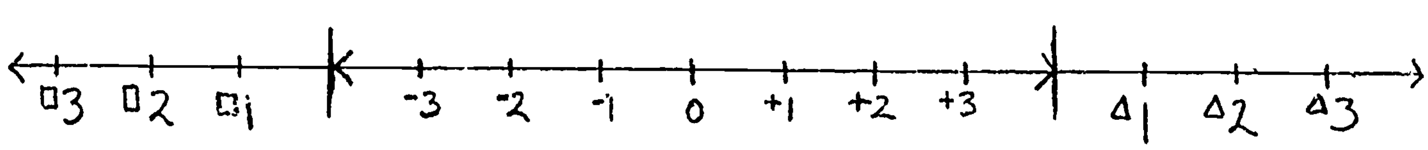


I tried to get the idea across that, whereas we can't write all the numbers down, we think of them as already being on the line anyway, and that consequently she had to tell me where $^{+}4$, $^{+}5$, and so on, all go without moving $^{+}3$. She replied that they all went to the left of $^{+}3$. I asked if they went past the vertical line she had drawn. She said no. I asked if she could come up and put an arrowhead on the line that would point to the rest of the positive integers. The finished product was the following number line:



I will never be certain, but it is my belief that she discovered for herself what is known in set theory as the order type $\omega^* + \omega^*$ (an ordering of this type would consist of two disjoint sets, both ordered just as the negative integers are ordered. Furthermore, every element of the first set would be regarded as being less than every element of the second set).

Whatever Leslie's original idea had been, I proceeded to develop the above interpretation of it with the whole group. She would have protested, if I had gone against her original intuition, and she did not protest. We concatenated the positive and negative integers (and copies of them) in various orders. The following number line typifies the ones we examined:



The big vertical slashes were called "fences." We read "$^{\square}1$" as "square one" and "$^{\Delta}1$" as "triangle one." It was understood that these latter were "new numbers." It is important to realize exactly how the children understood this number line. They knew that the truth set of $\square < {}^{\Delta}2$ contained, among other numbers: $^{\Delta}1$, $^{+}100$, $^{-}5$ and $^{\square}3$. We had already discussed the problem whether there is a last number, before Leslie created her non-standard number line. The issue was by no means settled. However, the class had some notion to the effect that if one began counting $^{+}1$, $^{+}2$, $^{+}3$,..., the process would go on forever. Thus they knew that if one began counting $^{+}1$, $^{+}2$, $^{+}3$, and continued in that fashion, one would never get past the fence separating "triangle land" from "positive-negative" land. I believe that most children accepted the idea that there were infinitely many numbers between the two fences.

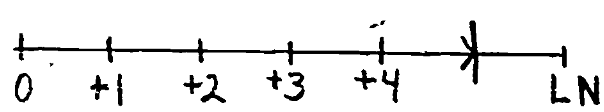
As will be seen, this class also came to understand what an infinite set is, as well as what an infinite cardinal number is. Their term for the cardinality of an infinite set was: the forever number (FN). I told them that "infinity" was another word for it. (Naturally we did not consider the possibility that there is more than one infinite cardinal.) One day while reviewing the above number line, I again asked if it were possible to get over the fence into triangle land by counting $^{+}1$, $^{+}2$, $^{+}3$,... Several children reaffirmed that it was impossible, whereupon one boy suggested: "Hop over it." When I asked how many numbers one would have to hop over, he said 100. Somebody else said 1,000. Then they all realized again that it was the forever number.

The non-standard number lines appealed strongly to their imagination. They once tried figuring out how to add numbers on a number line such as the previous one. Unfortunately that discussion got sidetracked. Also I forgot to ask them if $^{\Delta}1$ had an immediate predecessor. They would have figured out the correct answer. When I introduced $^{+}\infty$ and $^{-}\infty$ to the sixth graders, I asked them if any one number came just before $^{+}\infty$. It took them twenty minutes to come to the conclusion that there was none. Many said $\infty - 1$, but we had already developed a method

for performing certain algebraic operations with ∞ (some of the time), and according to this method one would have to conclude that $\infty - 1 = \infty$, so that $\infty - 1$ does not precede ∞.

Infinity was a popular concept in each class, and it was more popular in the second grade than in any other. The third grade was fascinated with the question whether there is a last number. In both classes pupils would call out a large number hoping that it might be the last number. I would always say that I could find a bigger one, and it didn't really matter which number they gave me. Then I would ask once again if they thought there was a last number. In both classes they hit upon the idea of adding one to or doubling any number that purported to be the last number. In the third grade they gradually accepted the notion that there was no last number.

In the second grade Kevin continued to insist that there was a last number. Finally (after Leslie had introduced her non-standard number line), I suggested that we could have a system where there was a last number, and I put the following number line on the blackboard:



LN stood for the last number. Then I said that we had a last number on the number line and asked if one could ever get to it by counting positive numbers. They agreed it was impossible. When I asked Kevin if there were a last number among the positive numbers, he said "No." He said it as if the present version was what he had intended all along. By looking at the question in terms of number lines, the class saw that many systems were possible, that a system might or might not have a last number, and that if one were lacking, one could add on a last number. We even briefly considered the idea of adding another number to LN, calling it LN + 1, and placing it to the right of LN on the above line.

I introduced notation for sets early at all grade levels, in order to be able to write truth sets and false sets. About the beginning of the second semester I took up the subject for the sake of certain related mathematical content. There is a tendency to view the study of sets in " new math" as artificial.

Indeed, set terminology could be regarded as superficial in many contexts, but set theory itself is part of the foundation of mathematics, so it is important to learn about sets. Nevertheless, at the beginning stages it seems to be a good idea for the teacher to come up with some interesting mathematics in which sets play a role, rather than merely to introduce some of the formalities. In particular, one should seek to find questions which require reasoning from the children.

The first idea to get across is the idea by virtue of which the sets $\{0,0,1\}$, $\{0,1\}$ and $\{1,0\}$ are all identical. This idea may be expressed by saying that two sets are equal exactly when they contain the same members. To test the children, I would ask how many members the set $\{0,1,a,0,{}^{-}1,1\}$ contained. The second and third graders were especially proud when they figured out the trick.

Another test consisted in asking them how many ways the set $\{a,b,c\}$ could be written, if no letter were written more than once. The children would suggest different ways of writing this set, and we would count how many had been obtained. The problem is to know when one is done. Eventually six ways were counted, and they were convinced that there were no more. The best reason they could give was that if you write it another way, it will be the same as one of the ways already written. Then one boy said: "There's six, because three times two is six." I tried to get him to explain what he was thinking. He more or less said that it was because there are three letters, and " if you start with a, then you can have either b then c, or c then b." He didn't completely finish, but he knew the correct way of seeing this problem. I had to be careful not to structure the pupil's thinking too much in these problems, for the pattern is a simple and beautiful one which they should discover for themselves. That way interest does not lag. In each class we worked on the following chart:

number of elements	number of ways of writing
1	
2	
3	6
4	

In effect the problem is: How many permutations are there of an n-element set? The answer is n! All pupils seemed to understand that it did not matter which n-element set one chose to examine. In fact, after the second graders had established that the set $\{a,b,c\}$ can be written six different ways, I asked how many ways $\{0,1,2\}$ could be written. One girl, Sandra, another bright second grader, instantly said, "Six!" I asked why, and she said, "Because $\{a,b,c\}$ can be written six ways."

The second graders never completely analyzed the problem for 4-element sets. A few of the third graders wrote out all 24 ways of permuting a 4-element set (as homework). One boy knew the correct way of grouping different permutations so as to reduce the problem to the problem for 3-element sets. He was eager to get to work on a 5-element set, and he got a good start.

The fourth and sixth graders were able to fill in the chart down to the seventh row in a matter of minutes. In each case they noticed the pattern of the numbers --- a new entry is obtained by multiplying the row number by the preceding entry. They realized that they had to check each entry by actually counting the permutations to see if the pattern they discovered was borne out in reality. Then I added that it was all right to follow the pattern, if they could give a reason for it. The sixth graders actually saw why the number of permutations of a 5-element set is 5 times the number of permutations of a 4-element set. The method of grouping the permutations according to the first letter was well enough understood that most sixth graders were able to see why the pattern is correct. The fourth graders had more difficulty. Only one or two saw it. The sixth graders understood letter variables. so we took advantage of the problem to define the function n!

In a similar fashion all classes but the third grade tried to calculate the number of subsets of an n-element set. The second graders solved it by brute force for the cases n = 0, 1, 2, 3 and 4. The fourth graders discovered the pattern, but they never saw the reason. In the sixth grade they got the pattern, and we were able to justify it by showing why the

number of subsets doubled from one case to the next (they saw it in passing from the case $n = 4$ to the next). Finally I asked the sixth graders to give me an algebraic expression involving n such that it would assume the values 2, 4, 8, 16, . . . They discovered that 2^n works (after trying $2n$, n^2 and several others). Later on in the sixth grade we studied sequences for their own sake. They became fairly proficient at discovering the pattern and finding a general term.

In the second grade I introduced the "how-many numbers," the cardinal numbers. A number was a how-many number, if it was the answer to the question how many elements a set contained. Thus, zero was the how-many number for the set $\emptyset$, the empty set, three for the set $\{0,1,2\}$, and FN, the forever number, for any infinite set. Dots were used to denote infinite sets such as $\{^+1, ^+2, ^+3, \ldots\}$ and $\{\ldots, ^-3, ^-2, ^-1, 0, ^+1, ^+2, ^+3, \ldots\}$. They saw that one had to be careful that the few numbers written established a pattern. For example, $^+8$ is an element of the set $\{^+2, ^+4, ^+6, \ldots\}$, but it is unclear whether $^+8$ belongs to the set $\{^-4, ^+3, ^+1, \ldots\}$. (The notion of pattern requires a stricter formulation, but only the sixth graders were ready for that.) The second graders were in the habit of writing the cardinality of a set below the set.

My plan was to introduce cardinal addition and apply it to infinite sets. Thus, we considered the set operations union and intersection. To introduce union, I put the following on the board:

$$\{^+1, ^+2\} \cup \{^+3, ^-1\} =$$
$$\{^-1, ^+5, x\} \cup \{0, ^-1\} =$$

and several other such incomplete sentences. I pointed out that we had a new symbol "$\cup$" and told them how to read the above. I said that $\cup$ was an operation on sets, a way of combining two sets to get a new set. I challenged them to figure out what the new set was in each case. Then I completed the first equation. Several already knew what the next answer was. Nevertheless, I did one more. I asked for volunteers to answer the third one, and they just about leaped out of their seats. Before any explanations were given, a great number of these prob-

lems were solved. Each time an answer was given, I simply said, " That's right" or " That's wrong." They would correct each other. They realized that the answer to the second problem could be either $\{{}^-1, {}^+5, x, 0, {}^-1\}$ or $\{{}^-1, {}^+5, x, 0\}$, because the two are equal. When I asked for the rule they were using, they had no trouble telling me.

Intersection was learned similarly, although it took longer. Then I gave them the word 'disjoint' and put sets on the board in pairs. Pointing to a pair, I would indicate whether those two sets were disjoint (without giving any other indication of the meaning of the term). After a while quite a few were able to specify whether two sets are disjoint. Then I asked how one could tell. For a long time the best answer they gave was: " When they're the same." With some assistance they formulated the idea that the same element couldn't be in both sets. Graphically the fact that two sets are not disjoint was expressed as follows:

$$\{{}^+4, {}^-3, {}^+1, {}^-3\} \quad \{0, {}^+1, 0, a\}.$$

The next day I returned to union problems. They noticed that if the cardinalities of the sets left of the equality were written below each set, the cardinality of their union could sometimes be obtained by adding the two numbers, other times not. We distinguished cases by calling them "CAN ADD" problems and 'CANNOT ADD" problems. They had to discover what makes a problem a can add problem. For quite a while they were puzzled. As we did more and more problems, they were able to make correct predictions. I asked, 'What is it about the two sets that makes a can add problem?" They puzzled some more, and we worked more examples. I repeated the question. This time Sandra responded, "When they're disjoint!"

I believe that they had long since seen that fact, so that the only remaining difficulty had been one of words. In many cases I found that they had a clear grasp of situations which they could not express in words. The degree to which the children should be pressed for words depends on the degree of involvement. If interest lags, it is better to ask questions which can be answered by a number or selection of one out of

several alternatives (two is too few, since the children guess, and one is forced to ask "Why?"). Usually when a child is sure, he will volunteer an explanation. Fortunately interest rarely lagged in the second grade.

I then explained how one could add how-many numbers using unions of disjoint sets. I hoped that they would learn that FN = FN + 1 and FN = FN + FN. The prospects were bright, for whenever an infinite set was show them in any guise and I asked them how many numbers were in it, they eagerly replied, "The forever number" or "infinity." If I would ask how many ways the set $\{0\}$ can be written (repetitions allowed), they would shout, "The forever number!" Whereas they were able to formally conclude that FN = FN + 1 by looking at the equation:

$$\{0\} \cup \{^+1, ^+2, ^+3, \ldots\} = \{0, ^+1, ^+2, ^+3, \ldots\},$$

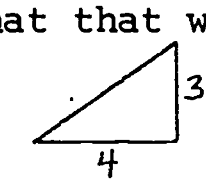
I do not feel that they really grasped what they were doing. For one thing, their (considerable) powers had been taxed with much complexity in developing the basic set notions as above. In retrospect I also suspect that FN did not have full status to them as a number, at least as a number that was eligible to be added to other numbers. I think Kevin, the brightest pupil, understood, but he had also been the one so eager to compute LN + 1 when we studied number lines.

One other major topic was discussed in the primary grades: area. In both the second and third grades I began by placing a line on the board (about three inches) and saying that I intended to call it one uni. (of length). Then I would draw a square with sides one unit long and call it a unit square. After that I drew a rectangle which was four units by five units. (We had learned the names of simple geometric figures.) I asked how many unit squares were in the rectangle. Various guesses were made. I asked someone to draw them in --- it took considerable time to measure carefully and get everything right. Mistakes were made. For example, one boy drew in five rows of (purported) unit squares with five squares per row. However, others corrected him. They could see that five unit squares cannot be placed along a line four units long. They decided that

2

20 unit squares could fit in. I said that that meant the area of the rectangle was 20 unit squares.

Each day I varied the unit of length. One day I made it one yard long and drew a one by two rectangle and asked for its area. They were convinced that its area was huge (over 100 unit squares). I said they were thinking of the unit of the previous day and reminded them that this rectangle was only two units wide and one unit tall. They reluctantly concluded that the rectangle had area two unit squares. Each child was given a lot of practice computing areas of rectangles. They all became adept at finding areas on graph paper. Many developed the technique of counting squares row by row. For example, they would count the squares in a five by six rectangle: "6, 12, 18, 24, 30." I encouraged such innovation. Kevin had heard of multiplication and realized that that was what he was doing. Then I drew a right triangle:



, and said that it was a three by four triangle. I extended it to a three by four rectangle using dotted lines. I drew in the unit squares and asked them to count how many were in the original triangle. At first they counted only whole squares completely inside the triangle. I suggested that some pieces in the triangle could be combined to make a whole unit square. They began to connect pieces at the blackboard. The fraction of a whole square which a piece may be can vary quite a bit, even in simple examples. Thus, they made mistakes and had to be cautious. Finally they counted six unit squares in this fashion. I said that was the correct answer and that I knew, because I had another way.

They worked more examples by piecing parts of squares together. As they became curious about "the other way," I suggested they look at the whole rectangle and think about its area. After a while several pupils could give correct areas of right triangles without counting squares. They had trouble formulating the reasons. Also I had to actually draw in the whole rectangle. At first the reason given that a three by six right triangle should have area nine was that nine is half of eighteen.

They were calculating the area of the whole rectangle. The question remained why half of that was the correct area. I had to ask what part of the rectangle the triangle was. They did not understand the question very well, not really knowing about fractions. Nevertheless, someone replied that it was half. I asked how they could tell that the triangle was half of the rectangle. Somebody said that there was another triangle that was half. I asked why one triangle wasn't more than half, and the other less than half. When someone said that the two triangles were the same, they agreed that that was the reason. However, they were pretty confused by so many hard questions. I believe that about three or four pupils followed the "proof."

The third-grade experience with area was slightly different, but they also learned to find areas of rectangles and right triangles.

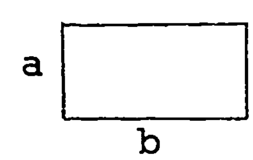
The remainder of the discussion of the mathematics covered during my teaching applies only to the upper grades. The major topics studied by the second and third graders have all been presented. The topics yet to be discussed were too advanced for the primary grade children, not because these children were too immature for them, but because adequate arithmetical (and also physical) skills had not yet been developed.

I stand in awe of the second graders. Not only could they debate sophisticated, abstract issues, but also they loved it, as their response and the regular teacher's comments to me indicate. Compared with the phenomenal second graders, the third graders were less impressive, but they did solid work above grade level in several areas. They became very proficient in adding signed numbers. They made considerable progress in learning to solve linear equations in one unknown. They learned to graph the truth sets of equations with two variables. They studied the order structure of the integers. They grasped the concept of infinity. They were able to compute areas of rectangles (even of figures obtained by pasting rectangles together) and of triangles. On many occasions they showed the same creativity as the second graders. The problem seemed to lie

in following a train of thought as a group. As a result, many fine ideas discovered by one pupil were lost on the others. It should be pointed out that I am applying very high standards. Every visitor to the class, including Johntz, who has had extensive experience with discovery classes, was highly impressed with both the work and attentiveness of the class.

My main goal in the upper grades, especially the sixth, was to lay a solid foundation for the study of (what is called) algebra in high school. We have already seen how both classes made considerable progress with linear equations (in the open sentence form). The best sixth graders eventually solved the problem in general. Moreover, the fourth graders made a partial analysis of the analytic geometry of linear equations. In order to familiarize them with adult terminology, I introduced the notion of letter variables. Historically this notion has been widely misunderstood. A variable is not "something which varies." It is a symbol. The idea of variation is embodied in the way one considers the different assignments of values in some set to a variable. The boxes and eggs are variables, and the rules for finding truth sets of open sentences are designed so that misconceptions as to what a variable is are avoided. Nevertheless, adults do not use boxes and eggs in algebra, so at some point a transition from boxes and eggs to letter variables must be made.

Early attempts in the sixth grade have been mentioned. On one occasion I put the following on the board:



I asked the class to suppose that a and b stand for numbers which give the dismensions of the rectangle. I said, "If I want to write the area inside the rectangle, what should I write?" It was a good question. They thought for a long time and made several mistakes before someone replied, 'a times b.'' I entered "a x b'' in the rectangle and told them it could also be written ''ab'' or '(a)(b)." We tried other examples using other letters or else one letter and one number. Moreover, I asked "What if a stood for 5 and b stood for 3? What would the area be?" They had trouble with such questions, for

whenever I gave dimensions inconsistent with the picture, they would say that the picture was wrong. I explained that I really only wanted the area given the sides and that the picture was just to help them think about rectangles.

Another example, due to Johntz, which I used in the fourth grade was the following: $\square + \square = {}^{+}3 \;\{N\}$. I would ask what number N stood for, or in any case, what were two numbers that N is between. They could see that N had to be between ${}^{+}1$ and ${}^{+}2$. They did not know much about adding fractions, but somebody did suggest that N stood for ${}^{+}1\frac{1}{2}$, and we were able to add ${}^{+}1\frac{1}{2}$ to itself by first adding the ones, then the halves and combining the results.

It was not satisfactory merely to speak of letters as "standing for numbers." Eventually a correct and more systematic treatment of letter variables was required. By the time I took up the matter, the sixth graders had had considerable exposure to letter variables, so I first tried the following approach with the fourth graders, who had had almost no exposure.

I explained that we were going to talk about strings of mathematical symbols and letters, that we would call them "letter expressions," that some letter expressions made sense, and others, such as "((+a" or "(=+a," did not, and that we would try to talk only about those which made sense. Without further explanation I said that we would also talk about substitution (of numbers for letters and letters for letters), and I wrote on the board:

$$a + b \qquad \begin{array}{l} a{:}5 \\ b{:}6 \end{array}$$

That meant that "a + b" was a letter expression and that we wished to substitute simultaneously "5" for a and "6" for b. I asked for volunteers to come to the board and write the result of performing the substitution just below "a + b." After the instructions were repeated two or three times, someone came to the board and wrote "5 + 6" just below "a + b."

They learned what to do very quickly. They also learned that "a:5" means that "5" should be substituted at <u>all</u> occurrences of a in a given letter expression. The problem

has interesting variations. I wrote on the board:

$$\begin{array}{ll} b \times (a + b) = a + (1 + c) & a: \\ 3 \times (2 + 3) = 2 + (1 + 2) & b: \\ & c: \end{array}$$

and asked them to indicate what substitution was performed. They figured it out. They learned that " $3 \times (2 + 3) = 3 + (1 + 4)$ " was not a substitution instance of the foregoing letter expression. They learned to find a letter expression, given the substitution. They learned that the answer to that question was not always unique. They learned how to find a letter expression and a substitution yielding a given expression.

We also considered substitution of letters for letters. One girl asked, " Can you substitute expressions for letters?" She also wanted to know if one could substitute letters for numbers. I told her that I was pleased she had thought of the possibility, that such substitutions were possible, but that we only would be doing the kind already described. I asked what was the result of substituting y for x and x for y in the expression " x + y." They replied, " y + x." Then I proposed doing it step by step. The problem was to see what was obtained if first y is substituted for x and then x is substituted for y in the result of the first substitution. At first they maintained that " y + x" was the answer. They had trouble understanding what I meant by " the result of the first substitution." I devised the following format to help them:

$$\begin{array}{lll} x + y & x{:}y & \\ \hline & & \text{FIRST STEP} \\ \boxed{} & \underline{y{:}x} & \text{SECOND STEP} \\ \boxed{} & & \end{array}$$

I explained that the result of performing the first substitution went in the first box and the result of the second in the second box. Several pupils still tried to perform the second substitution by referring to the top letter expression, but after a while everyone saw that " x + x " was the final result.

The next question was, " What if you switch the order? First substitute x for y, then y for x. What is the result?" They saw that " y + y " is obtained. We practiced with other

examples. I believe that a few of the fourth graders realized that simultaneous substitution cannot be performed " step by step." Finally I asked if it were possible to obtain " y + x" from " x + y" in several steps by performing only one substitution at each step. It did not occur to them to introduce a third letter (to avoid " collision of variables"). On that basis they were correct in concluding that it was impossible. I showed them how to do it in three steps using a third letter.

The exercises with substitution were interesting in their own right. They provided simple, thought-provoking questions. However, their purpose was to underscore the role of substitution in the following development.

I explained that some letter expressions were called letter sentences. A letter expression is a letter sentence if an ordinary closed sentence is obtained when numbers are substituted for the letters. The possibility of numbers occurring in letter sentences was emphasized also. We examined a variety of letter sentences. Then I said that we would call a letter sentence " always true" (AT), " sometimes true, sometimes false" (S), or " always false" (AF) in the following respective cases:

1) The result of substituting numbers for all the letters in the sentence is always true,
2) The result of substituting numbers for the letters is sometimes true and sometimes false,
3) The result of substituting numbers for the letters is always false.

I explained that AT-sentences were also called " laws of algebra," and that if they were equations,[x] they were also called " identities." In the sixth grade we did not discuss substitution systematically, but the foregoing definition of AT, S and AF was given informally in that class.

The task was to examine letter sentences and decide whether they were AT, S or AF. The fourth graders realized that " $a = a$," " $a + b = b + a$," " $a \leq a$," and other simple sentences are AT.

x One of the virtues of this approach is that the misleading practice of calling a sentence such as " $a + b = b + a$" a true sentence is avoided. It is an AT-sentence.

We always began by performing a few substitutions until it was clear. However, I would ask if it were enough to do a few substitutions. Several pupils understood that some reason was required, after I stressed this point. It was not always easy for them to formulate correct reasons, but the difficulty appeared to lie with the words, not with the concepts. I tried to get them to use the right words and repeat them aloud as a group. At first they were certain that " $a + a = a$ " was an AF-sentence, since several examples bore them out. I kept asking if they were sure. The brightest boy, Vernon, came up with the substitution $a = 0$ (eventually we slipped into language such as " let $a = 0$, " but only after they had a good understanding of AT, S and AF). It took quite a while to discover an AF-sentence which is an equation. They saw that " $a \neq a$, " " $a < a$, " " $a + 1 < a$, " and several others are AF. Then Vernon suggested " $2 + a + b = b + a$. " The class also learned that a special reason has to be given to justify calling a sentence AF, but that two substitutions, one yielding a true sentence, the other a false one, suffice to show that a sentence is S.

I introduced the sign " $^-$ " as a unary operation symbol for the additive inverse operation. The fourth graders learned how to use it, just as the sixth graders had. They had some difficulty seeing that " $a + {}^-a = 0$ " and " ${}^-(a + b) = {}^-a + {}^-b$ " are identities. In the sixth grade I wrote " ${}^-(a + b) =$ " on the board and asked them to complete it, so that we obtained an identity. They suggested " ${}^-(b + a)$, " which was correct, but not what I was hoping for. When I supplied the answer " ${}^-a + {}^-b$, " they had to check several examples before they were satisfied. On the other hand, it was immediately clear to them that " $a + {}^-a$ " should be set equal to zero.

In both upper grades subtraction had been discussed. They had seen, for example that ${}^+5 - {}^-3$ can be computed by asking what must be added to ${}^-3$ to obtain ${}^+5$, i.e., by finding the truth set of ${}^-3 + \square = {}^+5$. Both classes had difficulty with arbitrary subtraction, but the better students learned it. Thus, another valuable problem was the following: " Fill in the blank in '$a - b = a +$ ____,' so that an identity is obtained."

Actually this problem stumps many people who have had high school algebra.

Interesting problems involving the additive inverse operation are ones such as: "Let $x = {}^{-}y$, ${}^{-}a = y$, ${}^{--}b = a$, $c = {}^{-}b$, ${}^{-}d = {}^{-}c$, $e = {}^{-}d$, and $e = {}^{+}16$. What is x?" Such problems are like puzzles, and the children enjoyed them. By introducing addition and even multiplication, one can make them even more complex.

Both classes became quite familiar with letter expressions in algebra. Most of them were able to interpret complicated expressions correctly. Several fourth graders had trouble, and their interest lagged. As has been indicated earlier, about half of the sixth graders were not focusing their attention on the material, at least not enough to pursue the ideas very far. They were quite bright. Many of the girls, who for the most part had been maintaining a detached attitude towards mathematics, would become interested in a particular problem. They would get quite excited and give a star performance for the day. One sixth grade girl who had not been paying attention for a while rushed up to me during a recess and asked me to check her paper. The next day she was thinking about something other than mathematics.

In any event, in both classes we went another step in the study of the algebra of the real number system. The plan was to isolate certain identities, call them axioms (the field axioms) and see how the axioms could be used to prove other identities. In the fourth grade we only got to the associative and commutative laws of multiplication and addition. They could recognize these laws (as well as remember their names). We applied them to open sentences, rather than letter sentences. They were able to see how an equivalent open sentence can be obtained from a given one by the application of an axiom. Although they were good at transforming open sentences into equivalent ones, they did not really see why I kept asking for an axiom to justify each step. They seemed more interested in finding the truth set. I do not feel that I have found any very exciting technique for teaching this aspect of algebra.

Probably a formal, combinatorial approach, according to which strict rules are observed, would make the problem of proving algebraic identities more like a game, and hence more fun. My very difficulty in this instance illustrates how the pedagogical problems of mathematical education are primarily mathematical problems. At least they are problems which can be solved only by someone with an excellent understanding of mathematics.

In the sixth grade the students learned all of the field axioms. In fact, I handed them out to each pupil in tabular form for easy reference. We used them in doing proofs, but the sixth graders also preferred not to stress the details, just as did the fourth graders. Several were quite adept at performing the usual algebraic transformations, and they could justify each step with the appropriate axiom. I used the distributive law to prove that multiplication of positive and negative numbers obeys the rules they had already learned by rote. Only the best students followed the proof, although I have seen fourth graders learn it in a discovery class.

In order to show that the field axioms can apply to other algebraic systems, I introduced the field of integers modulo 5. They learned to add and multiply modulo 5 by watching me do a few examples (much as the second graders learned the union and intersection operations on sets). Quickly they figured out all the possible cases, and we entered the results in a table. They had to learn how to use the table, but when we compared it with the ordinary multiplication table, they had no trouble. In fact, I pointed out that tables could be used to define operations. I asked them if there were any feature of the table by virtue of which they could tell that the corresponding operation is commutative. By considering several tables, they were able to conclude that the operation is commutative when the table is the same above and below the diagonal, which I drew in. We saw that addition and multiplication modulo 5 are commutative. We were unable to ascertain finally whether the other field axioms hold in this system, for the end of the year was approaching, and I wanted to work on something else.

Up to now I have made little reference to a topic which proved very successful with the sixth graders, for the other groups did not cover this topic: infinite sequences. Early in February we were discussing what an infinite set was. They suffered from the misconception that a reasonable solution to the problem of finding the truth set of $\square = \square$ is to answer, " It's infinite!" I pointed out that such an answer applied equally well to the open sentence $\square > 5$, which clearly had a different truth set. To clarify the matter, I asked them to tell me what an infinite set was and to give me two different infinite sets. We concluded that an infinite set was one which " goes on forever," i.e., if one begins to count its members, the process would go on forever. Somebody asked if infinity were a number. I explained that one could take infinity to be a number, but it would not be in R, the set of real numbers. In fact, one could introduce $^{+}\infty$ and $^{-}\infty$. They were jealous when I related that my second graders were using $^{+}\infty$ as a number and doing very well. Somebody wanted to know if one can add and subtract ∞. Someone else asked what $\infty + 1$ was. At first I tried to discuss the problem using cardinal addition, but it didn't go over too well.

The next day I asked them how many numbers were between 0 and 1. At first they said none. I said that was wrong. Finally someone thought of 1/2. I asked for a number between 0 and 1/2. The answer was 1/4. Very soon we had constructed the sequence 1/2, 1/4, 1/8,... Then it occurred to me to ask what they thought $1/\infty$ might be. In effect, we concluded that $1/\infty$ should be 0 (if anything) on the basis that when the denominators in the sequence got bigger and bigger, they were, loosely speaking, approaching ∞, whereas the terms in the sequence approached 0. In my mind I had decided to let algebraic expressions involving ∞, $\infty + \infty$ for example, be defined as $\lim_{n\to\infty} x_n + y_n$, whenever this limit was uniquely determined by two sequences x_n and y_n which both converged to ∞. I attempted to explain the method. We considered the expression ∞/∞. They offered the answers 1, ∞, and even 0. I wrote on the board:

/ , / , / , / ,...$\to$:∞/∞

and said that if they filled in numerators and denominators which would both go to infinity, and saw if the resulting terms of the sequence approached a definite number, it might be reasonable to take that number to be the value of ∞/∞. Someone suggested 100/100, 200/200, 300/300,... They concluded that $\infty/\infty = 1$. Then I said, " Can you make it come out differently by choosing different numerators and denominators?" They wanted to start with big numbers, so I said it could be done with 1, 2, 3,... in the denominators. They had to see that it was important to evaluate each term, so I drew downward arrows under each term of the sequence. Finally I hinted that they could fill in the numerators so that the values of the terms would approach 2. Then someone suggested 2, 4, 6,... for the numerators. That choice would appear to imply that $\infty/\infty = 2$. They were able to make the limit of the sequence come out to be various values. I asked them what they thought ∞/∞ should be. They said, " You can't tell," or " It could be anything." I explained that since it had to be one number, but that the method would reasonably allow it to be any one of many different numbers, it couldn't be defined. I said that in mathematics one referred to such expressions as " undefined." I gave them $\infty + 1$ as homework. Several students found the right answers, but they had some difficulty realizing that one must show that the value obtained is independent of the choice of sequences converging to ∞. They also had difficulty with $\infty - \infty$, which we worked in class.

Another question which absorbed them was: " Can you fit all the integers onto a line one yard long?" I had drawn such a line on the blackboard. We began with 0, which went in the middle. I asked where 1 should go. Somebody suggested to the right of 0, so I asked how far. " Halfway," was the reply. They still did not see how all the numbers could go on the line. I asked where 2 might go. Then they placed it halfway between 1 and the end of the line. After that they had no trouble fitting all the numbers on the line.

I hoped that we could discuss convergence of infinite sequences in general. I introduced the terms " convergent" and

"divergent" and explained them by means of examples. The possibility of sequences converging to $^+\infty$ and $^-\infty$ was not excluded. However, it was difficult for them to understand what I meant by a divergent sequence. I put up sequences such as 0,0,0,... , 1/2, 2/3, 3/4,... and 3/2, 4/3, 5/4,... They could tell me to what number these sequences converged. I did not pursue the study of convergence.

I became interested in showing how sequences are defined by means of a general term, which is usually an algebraic expression involving the letter n. The problem was: Given the general term, find the sequence. Given the sequence (in reality, just the first few terms), find the general term. Not only were these problems an important step in their learning to use letter variables, but also they represented an introduction to the function concept, as will be seen. Moreover, the entire class became quite absorbed in them.

The format I used arose out of the problem of finding the number of permutations of an n-element set. The solutions were given in the second two columns in a chart. To shorten the answers, I introduced factorials. Once 1!, 2! and 3! had been defined, they were able to give the values of 4!, 5!, and so on. In class they computed 8! Someone announced that he would do 10! I said that he didn't have to, but he brought it in the next day carefully worked out. Then I began to ask what n! was, if n = 3, if n = 5, and so on. I erased the headings of the chart for the permutation problem and let n and n! be the new headings:

n	n!
1	
2	
3	
4	
5	
6	

We filled in the chart. For several days we studied n! They learned how to understand the expression:

$$n! = (n)(n - 1)(n - 2) \;....\; (1).^{x}$$

x One boy realized on his own that this equation is not valid when n = 0.

Many had trouble understanding the dots and the fact that the factors are supposed to decrease to 1, as one reads from left to right. It was necessary to consider the situation for many specific values of n. For homework they were asked to find other expressions for $n!/(n - 2)!$, $(n)(n - 1)!$, and many similar examples. One of the girls figured out in class that $(m - 4)!/(m - 6)! = (m - 4)(m - 5)$.

I switched to other functions (with n as the independent variable), and we tabulated the sequence of values. For example, we examined n, $n + 1$, $n - 1$, $2n$, n^2, 2^n, n^n, $1/n$, $1 - 1/n$, among others. Also they considered the asymptotic behavior of the ones which approach infinity. I would ask, for example, " Which gets bigger faster: $n + 100$ or $2n$?" Not only did this example illustrate what is meant by the asymptotic behavior, it underscored the fact that one function can lag behind another initially and then race ahead. They became quite cautious.

Perhaps the most interesting part of this topic was the problem of finding a general term to fit a finite sequence of values. They had no trouble with 2, 3, 4,... ; 1, 4, 9,...; or 2, 4, 8,... They would first find a few more values following the pattern already begun. I let them do this simple exercise, until all were able to predict the next term. Then I asked for an algebraic expression involving n and having the same (initial) values. They had some trouble with 1/2, 2/3, 3/4,... and 2, 5, 10, 17,... They became skillful at observing the sequences of differences. Thus they could indefinitely extend many sequences determined by polynomials in n of degree 2 (and even 3), although they were not always able to find the polynomial.

They became quite absorbed in the following problem: "Into how many regions do n doubly-infinite lines in a plane divide that plane, providing that no two are parallel and no three have a common point of intersection?" They began drawing lines and counting. By the time they had counted regions for the cases n = 1, 2, 3, and 4, many of them saw a pattern in the numbers and were able to extend the sequence according to that

pattern. They realized that one had to verify whether the pattern they saw gave the correct answers. One had to be very careful in counting the regions determined by 5 lines. The picture is even more confusing with 6 lines. I told them what the polynomial was, and we checked it against the values already found. To verify that the pattern was correct, we had to see why the increase in the number of regions was n + 1, if one more line was added to a configuration of n lines. Quite a few of them followed the general argument and could see why the pattern was correct. Because of the algebraic difficulties, I did not show in general that the values of the polynomial increased according to the same pattern. Such problems as this one provided the children with an introduction to the ideas underlying mathematical induction.

The last one and one-half months with the fourth graders and the last two weeks with the sixth graders were devoted to a very fine programmed workbook, entitled Symmetry, by Jamesine Friend and Patrick Suppes (Stanford, 1965). I am grateful to the authors for making copies available to me for use in my upper classes. Since the style of teaching was not the discovery technique during this period (mostly I assisted the pupils individually, while they worked at their seats), I will not discuss what happened in detail. The book taught the children how to compute the groups of rotations and reflections of plane figures by first determining the lines of symmetry. At first they cut figures out of the book and literally performed the indicated rigid motions. As proficiency developed, they learned to visualize the effect of a given motion, and then of two motions composed. Ultimately they could fill in a group table, when presented with a figure. Although we did not get that far in the book, the last sections dealt with properties of group operations. My fourth graders were able to compute group tables accurately after working slowly and carefully through the book. The sixth graders spent two weeks on the book. One boy took it home and finished it in one night. Several others almost finished. They all enjoyed it, but many did not concentrate (they were concerned with graduating at

that time), and these students required my assistance in the simplest computations.

Having described all of the major topics taught and the children's response, I turn to matters relating to the regular teachers in my classes. On various occasions each teacher told me that the children frequently worked on their mathematics problems during other school periods. Many were very eager about algebra (their term for the mathematics classes). According to the second-grade teacher, the children could barely contain their enthusiasm. Her words were, "They just love it! They're so glad when you come!" Each teacher reported that the children enjoyed the mathematics and looked forward to it. The only exception occurred with the sixth grade during the first month. Because of the conditions mentioned at the beginning of this report, the class was initially hostile to me. I discussed the problem with the sixth-grade teacher on several occasions, until we worked out a plan for handing out firm discipline. The turning point came when the principal spoke to them one day. He explained that the algebra class would be taken away, if they continued to resist it. He attempted to persuade them of its value and concluded by asking each pupil if he wanted the class to stop. The next day I told them that we would switch the mathematics instruction to some other class, if they did not want it. Since the algebra class was a matter of great pride to them (other sixth grades at Lincoln were not so honored), they sobered up when faced with the possibility of losing it. They agreed that it made no sense to try to have algebra without no-talking rules. After the first month they settled down. Much later they indicated in various ways how important the class was to them. For example, if I missed a day, they would come up to me at the recess before class the following day and demand to know why I had been gone. It was my rule never to teach on days when a substitute teacher was there. Any trouble with the class because of poor substitutes tended to carry over to succeeding days. When I explained this reason to the sixth graders, they would promise adamantly that they would not create any problems. On the other hand, the

initial control problem had damaging effects which lasted all year. I always had to worry just a bit about control. I believe it kept me from treating certain subjects the best way. Moreover, it was not possible for me to give individual attention to the poorer students at the start. They got behind and dropped out mentally. A couple of them tended to be disruptive the entire year because of their natural resentment of the situation.

Each teacher reported various instances of children who had surprised them in the algebra class with their ability. When the regular teacher forms a negative opinion of a child, the effect is often that the child's performance is somehow hindered by that negative opinion itself. My teachers all had their expectations for many of their children raised. In the second grade two or three of the children in the slower half were among the best contributors, and the teacher noticed it. One girl in the sixth grade had terrible home conditions. She was a serious problem in school and may have nearly insurmountable difficulties as she progresses. However, she was brilliant in the algebra class, whenever she tried. She could do as well as the best of the boys, although she did not follow the development as closely as they did. The esteem she received from me and consequently from the regular teacher must have been apparent to her. Each class contained similar examples.

Another significant feature of my teaching related to the regular teachers was that, with the exception of the second-grade teacher, they all tried to follow the mathematics. Needless to say, the sixth-grade teacher learned a great deal that must have been completely new to her. The third-grade teacher had studied a bit of college mathematics. She watched me very closely and employed some of my techniques (both pedagogical and mathematical) in her own teaching. She mentioned plans to take more mathematics courses from the university and become a mathematics specialist herself. The fourth grade teacher enjoyed learning most of the topics we covered.

To conclude the description of my teaching year, I will report on two one-day lessons which I taught at other schools

in Berkeley at the end of the year. I presented the introductory lesson on addition of signed numbers to a fourth-grade class in a school where the children come from highly educated, middle class families. Many of the fathers were university professors (including one Nobel Prize winner). The class was extremely alert and grasped the material immediately. Moreover, they enjoyed the mathematics immensely, as was evident. That judgment was later confirmed by various sources. Clearly such mathematics teaching as I presented at Lincoln would be successful at this school. One characteristic of the Lincoln children seemed to be missing in most of these children: a certain extra degree of spontaneity. It is not reasonable to draw conclusions from one experience. However, I have the feeling that a hesitancy to take wild guesses for fear of being wrong and an excessive desire to get the "right" answer, namely the one I wanted, might have placed certain bounds on the creativity of the class as a whole in a year-long mathematics program. They would have been extremely adept at grasping new concepts, but might have had trouble dreaming up their own.

I also taught a sixth-grade class at a school comparable to Lincoln. The lesson concerned prime numbers and prime factorizations. We had an interesting discussion on whether they could be certain that there are infinitely many primes. One boy suggested that we could always get more primes by taking a number such as 1,000,000 and adding small primes to it. When we examined 1,000,002, someone saw that it was divisible by 3. They finally concluded that they didn't have a good reason to feel as certain as they did that there are infinitely many primes. The discussion was very lively. Their teacher was outstanding, and they were the best sixth grade at their school. When I left the school, the teachers were asking me to arrange for discovery mathematics classes at their school the following year.

I will now attempt to evaluate the foregoing data with regard to the hypotheses, one by one.

1. I believe that the testimony of the regular classroom teachers indicates that interest was generally quite high all year. I found that it varied from time to time, depending usually on my choice of material and manner of presenting it. I have illustrated how each class responded with intense excitement at some points and with moderate interest at others. Yet overall motivation was so high that, after a brief lag, the children's excitement would be fully renewed once I had modified my approach appropriately. The second grade maintained a very high level of interest almost all the time. Their love for intellectual discussion was obvious, as was their ability to create new concepts by themselves.

In the third grade various topics absorbed the children, but the extra degree of attentiveness which was lacking kept them from probing as deeply into certain subjects as I would have liked. The size of the class first semester (33) hampered communication. It especially kept me from providing the children with sufficient individual attention. It is evidently the reason that many children's minds tended to wander. I simply could not talk to all 33 at once. When the class size dropped to 23 second semester, the situation improved. Interest was higher. Since I could spend more time with individuals, I became more effective. If class size had been 23 from the start, the entire year could have been as stimulating as it was with the second graders. I feel that the style and pace set at the outset have a strong influence on the entire development. The sixth-grade control problem is a case in point. Measured according to my original expectations, rather than with the performance of the second graders, the third grade maintained a good level of interest. I neglected to point out that they frequently requested homework problems, when I forgot to give any. Many children worked on them at home and supplied correct answers the next day. Such motivation is regarded as unusual among disadvantaged children.

The fourth graders maintained a constant high level of interest. It tapered off slightly around March, but I introduced the workbook on symmetry and groups at the end of March. The

change of pace renewed it. For about two days at the end of the year I returned to the discovery method, applied it to what they had learned about groups, and they were fresher than ever. There were four or five children in this class whose interest lagged. These children felt they were not as good as the rest of the class. It is possible that their teacher reinforced this feeling, for he told me once or twice, that they probably ought not to be in the class. I would have to blame myself for not having drawn them in with special questions.

About half of the sixth graders were at least as interested in the mathematics as the second graders. As I have indicated, the other half, which included most of the girls, was sporadic in its interest. They tended to flash on and off like light bulbs. There were about three "mental dropouts."

There were some indications from the teachers that interest in schoolwork increased for some children. In many cases, but not all, the reference was to arithmetic (the children had an arithmetic period in addition to the algebra). I suspect there were many more cases than I knew of and that the increased interest carried over to subjects such as reading and science. Whereas high interest and motivation existed in my children throughout the year, I am not really in a position to speak scientifically of an increase in the general level. I cannot conceive of any case other than that it was greatly increased relative to what it would have been, had the children had a typical school year without me, but I do not know of a "scientific" means within the scope of this one-man project of measuring the increase. The special prestige attached to "having algebra" among classmates from other rooms was felt by my students, but it was not a consequence of the teaching itself. Also my students were above average students at Lincoln to begin with, so any high quality work that they did or special interest shown could be regarded as a reflection of that fact rather than as evidence of an increase in interest. The best indication of an increase lies in the unanimous judgment of all who have viewed discovery mathematics classes. They find the level of interest and achievement far, far above all

expectations. This fact convinces me more than anything else that the first hypothesis is fully borne out by the data.

As a footnote to the previous remarks, it should be pointed out that we are really interested in long-term increase in motivation, whereby ultimately these students will go on to various forms of higher education. Such an increase could be measured more precisely in the context of a long-range study.

2. There is no question that the children learned some mathematics. In each case they learned material way above grade level, and in every class except the third grade they learned some quite sophisticated concepts which underlie various types of mathematics not normally taken up before college.

The second graders developed strong intuitions about infinity. They knew what was meant by an infinite set. They were on the verge of learning how to do cardinal addition with the infinite cardinal FN. I have no doubt but that had I found time to spiral back to the topic, they would have mastered it. Imagine children able to do addition problems with infinity before they learn to add numbers above 100. In a sense they were also familiar with infinite ordinals, or rather with some of the order types generated by ω and ω^* under addition of order types. They saw that there could be numbers on a number line to the right of a given number which can never be reached by starting with the given number and moving to the right one step at a time. They knew that infinitely many numbers would be between two such numbers.

The other mathematics which they learned was more like mathematics studied in the schools, but not in the elementary schools. The basic algebra of the integers (with addition and the order structure) which we developed was perhaps not difficult, but they understood it with greater clarity than most high school students. They had learned to enjoy looking for reasons. Most high school students find it a waste of time to give reasons. Other advanced topics which they learned included sets, set notation, and set operations, counting techniques, and area of rectangles and right triangles.

The third graders had a solid algebra course in which they

too gained a firmer foundation than most high school students. Part of the credit is due to the open sentence-truth set approach. This way of understanding what is meant by the solution of an equation is mathematically more sound than that evolving out of traditional techniques. However, their curiosity, not yet stifled by mathematical miseducation, as it has been stifled for so many high school students, was more significant in getting them to reason and discover. The discovery approach is designed to stimulate that curiosity.

I found that the fourth graders did well with an introduction to what might be termed the metamathematics of algebra. They studied formal expressions (in an algebraic language), the properties of formal substitution in these expressions, and a kind of semantics appropriate for this language. The comprehension of most pupils was good, and they were able to solve simple problems.

Another advanced subject to which they had a successful introduction is group theory. They knew how to compute the groups of rigid motions of plane figures. It should be mentioned that the discovery approach was not used when we studied the subject.

In addition the fourth graders did well in traditional algebra. They made great progress toward obtaining a general method for solving all linear equations in one unknown. They made great progress with the analytic geometry of linear equations in two unknowns.

The portion of the sixth grade which was trying all the time astounded me with its ability to handle advanced subjects. Although we shifted from one topic to another quite frequently, this class's brief exposure to the function concept, to the behavior of sequences in the limit, to the properties of the extended real number system, to mathematical induction, and to the field axioms was successful. They solved or understood problems in these areas, which were rather sophisticated. Just as my other classes did, they achieved a high level of understanding of traditional algebra. Several of them knew how to solve any linear equation in one unknown.

The conclusion that disadvantaged children can learn advanced mathematics is demonstrated by the successes I have described. I have applied a mathematician's judgment to the question of how well they learned it. I conclude that they learned it well enough to warrant stating that the second hypothesis is fully supported by the data.

However, the more advanced concepts were learned in a very intuitive and unsystematic fashion. In order for them to fully assimilate such concepts, they will need to meet them again and again. They will need to work on correct ways of expressing themselves mathematically, especially in writing. The job of following up the teaching they received last year must be done, or they will lose the flare they developed for dealing with sophisticated mathematical notions. If the job is done properly, I would expect them to become proficient in various areas of university mathematics before they leave high school.

It remains to deal briefly with the remaining hypotheses, which, as mentioned earlier, are rather specific conjectures about the effects of varying certain aspects of the teaching within the framework of the basic teaching situation.

3. My classes were homogeneously grouped by ability, except that there were two levels in the second grade. These levels were not really very far apart in mathematics, except that two or three children were way below the others. The teacher split them into two groups based on reading levels. For her convenience she seated them in two different halves of the room. To whatever extent there was lack of homogeneity, it did not hamper my teaching in that class. However, the regular teacher's control was so outstanding, that all children paid attention all of the time. In the other classes those few children who were significantly less able than their classmates tended to disrupt the class. Many of them learned little or none of the mathematics, because they could or would not participate at the same level as the others. Some turned to mischief. Since they were a minority, it didn't bother me.

Thus my actual experience justifies somewhat weakly the conclusion that homogeneity is needed. I feel strongly that it is very necessary. If I had taught a class with a wide spread of abilities, it would have been impossible for me to communicate with the whole group, and such communication is fundamental to the discovery approach. The students not paying attention, because my remarks would not have been geared to their level, would have formed a majority. The extent of the disruption and inattentiveness might have been catastrophic. I would not care to do the teaching necessary to demonstrate this hypothesis with absolute conclusiveness.

4. During the period when the sixth-grade teacher still lacked control, my teaching relationship with the children almost collapsed. Damage was done which kept me from ever reaching some pupils. I could communicate with others only on an off-and-on basis. Control in the second grade was ideal. So was the class. Control was good in the other two classes, and the mathematics instruction was quite successful.

5. The one course of action which kept me from giving up the sixth graders during the first month was the use of strict discipline in conjunction with the regular teacher. For several weeks many children had to stay a full hour after school and write compositions. The principal spoke to them firmly. They were told that they might lose the algebra class. These unpleasant measures did not seem to cause any resentment. (After a while the sixth graders were very friendly towards me.) About half of the class went on to become my best pupils, along with the second graders. Accordingly, the extent to which the class turned out well was due to the measures taken to establish control.

6. When some of my classes had as many as 29 pupils, I found myself under some strain in trying to include all of them in the discovery teaching. In the third grade, which originally had 33 pupils, I found that size kept me from being as successful as I had wished. When the size dropped to 23, I could communicate with all the class. My teaching and their learning improved. I would not like to see class size drop below 15,

because the discovery method thrives on a lot of group excitement. Fifteen children might not be able to generate enough excitement.

7. I have visited discovery mathematics classes (even a second grade) which maintained a high level of excitement for 50 minutes. In my own classes I do not feel that the ideas we discussed could have been developed adequately in fewer than 35 minutes.

8. It seemed to me that whenever I would give extended explanations or whenever I would "tell" students how to solve a problem, they would tend to lose interest and would not really understand. Whenever I was able without using too many words to get them to see that some problem really was a problem, they would eagerly look for the answer. The problem would only be solved when they had done it themselves. If I structured their thinking too much --- say by pushing them in a certain direction, they were in danger of never understanding the solution, even after it was presented. With each topic I would rack my brain for questions to ask, for the way to set up certain problems, and for the hints to drop so that the children would see what the problem was, while my involvement would be held to a minimum. This last hypothesis was borne out by my experience each day.

5. Conclusions and Implications

The results of the previous section indicate that the research problem (I) has been solved affirmatively to the extent possible in a one-year study based on the experiences of one teacher. If care is taken to ensure proper teaching conditions, discovery teaching of mathematics provides the most exciting and successful classroom learning I can imagine. Moreover, the capacity of young children, in particular culturally disadvantaged children, to learn sophisticated mathematics is quite high. All that is needed is a willing mathematician.

In closing I would like to discuss certain implications of these conclusions and ideas which they suggest. First of all, the conclusions warrant using discovery teaching of mathematics as an important compensatory education program. Its use should be widespread. Moreover, a pupil should enter such a special mathematics program when he starts school and leave it when he has finished high school. There is no other way to provide the follow-up of a given year's mathematical growth called for in the previous section. After two or three year's away from this program, a child might slip completely back into the anti-intellectual attitudes of his peers and lose most of what he had gained. If he stayed in the program until the last year or so of high school, he would enter college with an ideal mathematical background. Of course it remains to be tested whether the same high level of student enthusiasm would extend over a period of many years and help to carry a child through school and into college.

Widespread implementation of such a program may sound visionary, but Johntz is thinking in such terms. I believe he is right in doing so. Other educators ought to become acquainted with discovery teaching of mathematics. They ought to consider seriously the feasibility and consequences of widespread implementation of discovery mathematics programs

which cover grades K-12.

Let us briefly examine the feasibility and some of the possible consequences. As a growing number of school districts initiate and expand this program, the problem of finding enough qualified mathematics teachers will arise. This problem is by no means insurmountable. I indicated at the outset of the report that the discovery mathematics teacher ought to be at approximately the bachelor's degree level of training, or else at a higher level. In an area such as the San Francisco Bay Area, which includes Berkeley, there are several colleges and universities. Graduate students of mathematics, and even undergraduates in the Senior year, constitute an excellent supply of teachers. So far, each mathematics student who came to Johntz with a desire to teach in his program ended up being a successful discovery mathematics teacher. Students are also likely to be in need of financial support. Teaching children is a rewarding way to obtain the support for one's further studies.

Despite the large number of mathematicians available in the Berkeley area, there would not be enough to go around even in Berkeley alone, if some system of priorities were not used. Those schools where need and likelihood of success are greatest should have a discovery mathematics program. The school with the greatest need would be one where the level of schoolwork is most depressed. Likelihood of success exists at those schools (and school districts) where the administrators would have a positive attitude towards the program and would be eager to assist the mathematics specialists in creating teaching conditions favorable to the discovery method.[x]

Mathematics students are not the only source of discovery teachers. Johntz believes that someday many university professors will have dual teaching assignments: part of their teaching will be done at the university, part in the elementary schools. I feel that many college and university professors would enjoy teaching children and that it is not at all far-fetched to envisage professors teaching discovery mathematics classes.

x Berkeley and Richmond are fortunate in this regard.

Even in areas without a high concentration of colleges and universities there may be considerable wasted mathematical talent. Many people who earn a bachelor's degree in mathematics and do not earn a higher degree might prefer teaching discovery classes over the jobs in mathematics usually open to them.

No discussion of scientific resources in the United States can avoid dealing with the fact that there are a great many mathematically trained people working in war industries. The country's military strength is now so overwhelming, on the one hand, while its social problems are so dire, on the other hand, that we all have a responsibility to call for a major shift in emphasis. Without such a shift the quality of American life will deteriorate. It is time to convert to a peace economy and use our scientific resources to solve the society's social problems. One of the best ways to begin would be by reallocating mathematical manpower from military research to discovery mathematics programs. If such reallocation were supported by national policy, shortage of qualified mathematics teachers would not stand in the way of implementing discovery programs on a large scale.

If we accept that it is feasible to start discovery mathematics programs running from grades K through 12 in those schools where the priority is high, we must examine the possible consequences of doing so. If discovery mathematics teaching is as successful over a period of several years as it has been seen to be in one year, we have the key to breaking the disadvantaged out of the economic trap they are now in. Many children would complete a program providing them with one of society's most valuable intellectual assets: the ability to do mathematics. Whereas mathematics would have been a stumbling block to them in traditional curricula, it would become the vehicle carrying them through school and into college. Thus, a specific consequence of such discovery programs in mathematics would be that many more disadvantaged children would go to college than ever before.

The quality of whole schools would improve, if a sizeable group of children at a school were successfully dedicating their efforts to learning mathematics. Many bright youngsters who do not normally do well in school and who would not be highly thought of by their regular teachers, might be recognized by their discovery mathematics teachers as bright children who can star in mathematics. A good discovery class would be able to absorb several youngsters known to be problem children, but also suspected to possess creativity and hidden intellectual potential. In the mathematics program such children could be saved. Discovery mathematics teaching might also be introduced to below average classes as a means of giving them a boost and a sense of pride.

It is not the function of this report to pursue the long-run social consequences resulting when many children from disadvantaged backgrounds obtain the economic foothold in society that higher education provides. However, it would appear that discovery mathematics programs have an important role to play in reducing the social and economic inequities of our society.